COMBINING CHARACTERISTIC NUMBERS TO REDUCE THE UNCERTAINTY OF FRACTIONAL-ORDER PROCESS MODELS

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Abstract: The paper extends the authors' previous results where the class of all lag/dead time process models (with integer or fractional order poles) is considered in the case that the following characteristic numbers are given: a) one sample of a process frequency response, b) first three moments of the process impulse response. In both cases, the explicit parameterizations of the value set boundary were provided. In this paper, the simultaneous knowledge of the characteristic numbers a) and b) is assumed and the corresponding upper hard bound of the value set boundary is described. It is shown that combining experimental data a) and b) leads to the significant reduction of a model uncertainty in the frequency domain. Additionally, the investigated results are applied to the robust controller design.

Keywords: robust controller design, fractional order pole models, value set, frequency domain

1. INTRODUCTION

It is well known from the classical control theory (Åström and Hägglund (2001)) that process controllers can be designed on the base of a few points of the process frequency response or on the base of some rectangle pulse response. In this direction, the limit method is the popular Ziegler-Nichols method which uses only one so called ultimate point or two numbers obtained from the step response. However, these traditional methods are neither systematic nor guarantee fulfilment of design specifications for an exactly given class of process transfer functions (Schlegel (2002)). It was shown in previous papers Schlegel and Čech (2004), Schlegel and Čech (2005) that after adding the key *a priori* information about the process transfer function to experimental data one obtains closed areas - value sets - quantifying the amount of model uncertainty at a given frequency. Using these value sets, one can design the robust controller guaranteeing fulfilment of some design specifications (e.g. gain and phase margins) for all processes consistent with *a priori* information and experimental data. In accordance with the majority of works in process control field, it was assumed that the real process can be described by a multiple fractional order pole model Charef et al. (1992); Podlubny (1999) in the form

$$F(s) = \frac{K}{\prod_{i=1}^{p-1} (\tau_i s + 1)^{n_i} s^{n_p}}, \qquad (1)$$

where $K > 0$, p is an arbitrary integer and $\tau_i > 0$, $n_i > 0$, $i = 1, \ldots, p$ are real numbers. It is believed that the class of transfer functions (1) is sufficiently rich at least for control purposes because it includes also all integer order lag/dead time processes of arbitrary order. The main result of previous papers was the explicit parameterization of the value set boundary for the model family consistent with the *a priori* information (1) and a) one experimentally obtained point of frequency response b) first three moments of the process impulse response. The sample of the frequency response usually specifies the process dynamics at higher frequencies, but the model uncertainty at low frequencies is very big. The phase shift of the sample is usually chosen in the interval $\langle 90^o, 180^o \rangle$. On the contrary, the experimental data b) specify the model precisely at a frequency $\omega = 0$. The model uncertainty increases at higher frequencies which are critical for robust controller design. From the previous it follows that combining for example suitable chosen sample of the frequency response together with moments of the impulse response will be very usefull for controller design. In this paper, the simultaneous knowledge of experimental data a) and b) is considered. The aim of this paper is to discuss advantages/disadvantages of several combinations of experimental data a) and b) and to show that this combination reduces the model uncertainty. The authors believe that the presented results have important applications in design of robust controllers and automatic tuning procedures (Schlegel et al. (2000)). Note that the data a) and b) can be obtained by the two most common industrial experiments. To obtain the sample of frequency response, the relay identification experiment is usually used. The moments of the impulse response can be determined from the rectangle pulse response (Schlegel et al. (2000)).

The paper is organized as follows. In Section 2, the previous results are remembered. Section 3 shows that the combination of characteristic numbers leads to the significant reduction of model uncertainty in the frequency domain. In Section 4, the investigated results are applied to robust controller design. Section 5 contains some concluding remarks and ideas for further work.

2. PREVIOUS RESULTS

General frequency response interpolation problem

Consider the set of frequency response interpolation conditions

$$\Pi = \left\{ F^{(l)}(s)|_{s=\mathrm{j}\omega_i} = P_{il} \right\}, \tag{2}$$

where $i = 1, \ldots, k$, $l = 0, \ldots, m_i$; k and m_i are given integers and $P_{il} \in \mathbf{C}$. In other words, the frequency response samples with some derivatives at frequencies $\omega_1, \ldots, \omega_k$ are given.

Definition 1. (Process model set). The transfer function $F(s)$ is *unfalsified*, if following conditions are satisfied
(i) Transfer function $F(s)$ is in the form (1), $n_i < m, \forall i, \sum_{i=1}^{p} n_i < n$, where $n \in \mathbf{R}^+$ is the total order of the process and $m \in \mathbf{R}^+$ is the minimum order of each pole.
(ii) Transfer function $F(s)$ satisfies interpolation conditions (2).
The set of all unfalsified transfer functions will be called *process model set* and denoted by $S^{n,m}(\Pi)$.

Definition 2. (Value set). The set $V_\omega^{n,m}(\Pi) = \{F(\mathrm{j}\omega) : F(s) \in S^{n,m}(\Pi)\}$ in the complex plane will be called the *value set* of the model set $S^{n,m}(\Pi)$ at frequency $\omega > 0$.

Definition 3. (Ultimate transfer function). The unfalsified transfer function is called *ultimate*, if there exists $\omega \notin \{\omega_1, \ldots, \omega_k\}$ such that $F(\mathrm{j}\omega) \in \partial V_\omega^{n,m}(\Pi)$, where $\partial V_\omega^{n,m}(\Pi)$ denotes the boundary of the value set.

For robust controller design, it is sufficient to find the boundary $\partial V_\omega^{n,m}(\Pi)$ of value sets in the frequency domain. This boundary is created for each frequency $\omega > 0$ by ultimate transfer functions.

The previous papers deal with two special cases of the general frequency response interpolation problem, where the interpolation conditions Π on the transfer function $F(s)$ are in the following forms

First three moments of the impulse response

In Schlegel and Čech (2004), the interpolation condition is the sample at frequency $\omega = 0$ and first two derivatives at this frequency, thus

$$\Pi = \{F(0), F'(0), F''(0)\}. \tag{3}$$

As it was shown, these characteristic numbers can be computed from first three moments of the impulse response

$$m_i = \int_0^\infty t^i h(t) \mathrm{d}t, \quad i = 0, 1, 2, \tag{4}$$

which can be also equivalently substituted by another three numbers k, μ, σ^2. The relationship between three equivalent groups of characteristic numbers is given by

$$\begin{aligned} \kappa &= m_0 = F(0), \\ \mu &= \frac{m_1}{m_0} = -\frac{F'(0)}{F(0)}, \\ \sigma^2 &= \frac{m_2}{m_0} - \frac{m_1^2}{m_0^2} = \frac{F''(0)}{F(0)} - \frac{F'(0)^2}{F(0)^2}. \end{aligned} \tag{5}$$

The main Theorem of that paper gives an exact parameterization of all ultimate transfer functions for the case of interpolation condition (3). The Theorem claims, that all ultimate transfer functions $P \in S^{n,m}(\Pi)$ can be expressed in the form

$$P_1 = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n_1(\alpha)}\,(\tau_2(\alpha)s+1)^{n_2(\alpha)}}$$

or in the form

$$P_2 = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{m}\,(\tau_2(\alpha)s+1)^{m}\,(\tau_3(\alpha)s+1)^{n-2m}},$$

where the functions $K(\alpha)$, $\tau_1(\alpha)$, $\tau_2(\alpha)$, $\tau_3(\alpha)$, $n_1(\alpha)$ and $n_2(\alpha)$ are exactly specified in the paper mentioned above. The value set boundary is created by three or four smooth arcs.

One sample of the frequency response

In Schlegel and Čech (2005), it is assumed that one sample of the process frequency response was obtained at frequency ω and no derivatives of the frequency response at this point are known, thus

$$\Pi = \{F(\mathrm{j}\omega)\}\,. \tag{6}$$

The main Theorem of that paper gives an exact parameterization of all ultimate transfer functions for the case of interpolation condition (6). If the interpolating condition is rewrited into the form $\Pi = \{(p_1, \omega_1)\}$, $p_1 = r\mathrm{e}^{-\mathrm{j}\varphi}$, then one can briefly summarize, that all ultimate transfer functions $P \in S^{n,m}(\Pi)$ can be expressed in one of the following forms

$$(i)\ P_1(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n_1(\alpha)}}, \tag{7}$$

$$(ii)\ P_2(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n-m}(\tau_2(\alpha)s+1)^{m}}, \tag{8}$$

$$(iii)\ P_3(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n-n_2(\alpha)}s^{n_2(\alpha)}}, \tag{9}$$

where functions $K(\alpha), \tau_1(\alpha), \tau_2(\alpha)$ and $n_1(\alpha)$ are exactly specified. According to this theorem, the value set is for each frequency $\omega \neq \omega_1$ a closed area bounded by three smooth arcs (7-9).

3. COMBINATION OF CHARACTERISTIC NUMBERS

In this Section, the simultaneous knowledge of several types of characteristic numbers (3) and (6) is considered. Let the interpolating conditions for frequency ω_i be denoted by Π_i, $i = 1, 2, \ldots, k$. Firstly, one has to ask when the process model set will be non-empty or in other words when there exists at least one transfer function in the form (1) satisfying all interpolating conditions Π_i, $i = 1, 2, \ldots, k$ simultaneously.

Generally, a necessary condition can be formed. It is easy to prove that the process model set is empty, if there exists $\omega > 0$ such that $V_\omega^{n,m}(\Pi_1) \cap V_\omega^{n,m}(\Pi_2) \cap \ldots \cap V_\omega^{n,m}(\Pi_k) = \emptyset$. It means that the emptiness of the model set can be checked graphically by intersections of the value sets corresponding to the same frequency ω. Using these intersections, also the hard bounds of the value set $V_\omega^{n,m}(\Pi)$ can be determined because any point of $V_\omega^{n,m}(\Pi)$ must fall all into $V_\omega^{n,m}(\Pi_1)$, $V_\omega^{n,m}(\Pi_2), \ldots, V_\omega^{n,m}(\Pi_k)$ for any ω as follows directly form the definitions. Consequently, it holds that

$$V_\omega^{n,m}(\Pi) \subset V_\omega^{n,m}(\Pi_1) \cap \ldots \cap V_\omega^{n,m}(\Pi_k). \tag{10}$$

Since Theorems given in previous papers give an effective method for computing boundaries $\partial V_\omega^{n,m}(\Pi_i)$ for two special cases (3) and (6) mentioned above, the boundary of the intersection involved in (10) can be simply computed and the hard bounds of the value set $V_\omega^{n,m}(\Pi)$ determined. As the intersection of corresponding value sets is much smaller than the value sets itselves, the model set uncertainty is reduced significantly.

In the following, also the necessary and sufficient condition of the model set emptiness is given for some special cases of combination of characteristic numbers. Using this necessary and sufficient condition, it can be simply checked whether the experimental data Π are consistent with the *a priori* assumption on the process transfer function.

Two samples of the frequency response

The interpolating condition is considered in the form

$$\Pi = \{F(\mathrm{j}\omega_1), F(\mathrm{j}\omega_2)\} = \{\Pi_1, \Pi_2\}\,, \tag{11}$$

where $F(\mathrm{j}\omega_i) = p_i$. In this case, the model set is non-empty, iff $F(\mathrm{j}\omega_1) \in V_{\omega_1}^{n,m}(\Pi_2)$. In Fig. 1 and 2, two examples are shown with the samples phase shifts $90^o, 180^o$ and $180^o, 270^o$, respectively.

Remember that the suitable choice of the phase shifts of two samples p_1 and p_2 is necessary. If the samples are too close in frequency domain, the noise can cause the emptiness of the model set.

One sample of the frequency response and moments of the impulse response

The interpolating condition is considered in the form

$$\Pi = \{F(\mathrm{j}\omega_1); F(0), F'(0), F''(0)\} = \{\Pi_1; \Pi_2\}\,. \tag{12}$$

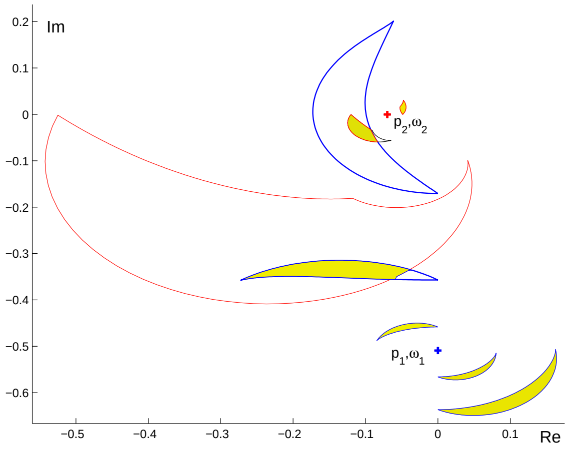

Fig. 1. Computing hard bounds of the value set $\mathcal{V}_\omega^{n,m}(\Pi)$ for the case, $p_1 = 0.5\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$, $\omega_1 = 0.30$, $p_2 = 0.07\mathrm{e}^{-\mathrm{j}\pi}$, $\omega_2 = 1.18$, $\omega = \{0.24, 0.27, 0.34, 0.40, 0.90, 1.41\}$; $V_\omega^{n,m}(\Pi_1)$ - blue line; $V_\omega^{n,m}(\Pi_2)$ - red line.

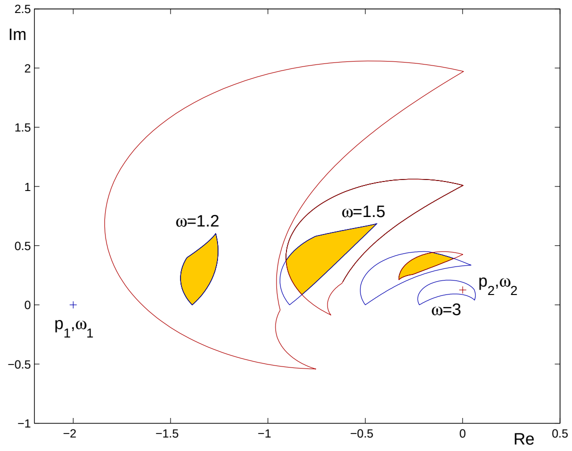

Fig. 2. Computing hard bounds of the value set $\mathcal{V}_\omega^{n,m}(\Pi)$ for the case, $p_1 = 2.0\mathrm{e}^{-\mathrm{j}\pi}$, $\omega_1 = 1$, $p_2 = 0.12\mathrm{e}^{-\mathrm{j}\frac{3\pi}{2}}$, $\omega_2 = 3$, $\omega = \{1.2, 1.5, 2, 3\}$; $V_\omega^{n,m}(\Pi_1)$ - blue line; $V_\omega^{n,m}(\Pi_2)$ - red line.

In this case, the model set is non-empty, iff $F(\mathrm{j}\omega_1) \in V_{\omega_1}^{n,m}(\Pi_2)$.

An example is shown in Fig. 3. This combinations of experimental data seems to be the most perspective one because of features mentioned in introduction. The right choice of the frequency response sample phase shift is usually about 135^o, where the uncertainty of the moment model set is maximal.

Two samples of the frequency response and moments of the impulse response

The interpolating condition is considered in the form

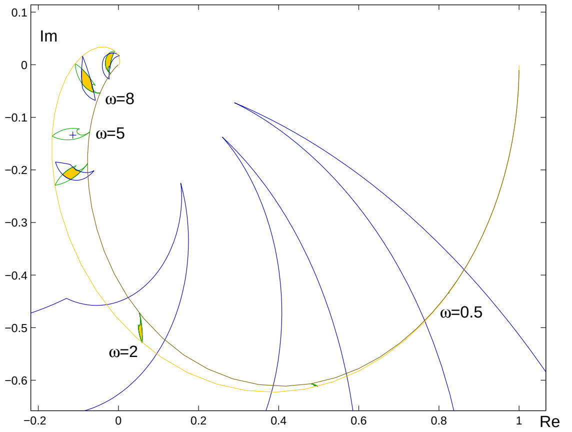

Fig. 3. Computing hard bounds of the value set $\mathcal{V}_\omega^{n,m}(\Pi)$ for the case, $p_1 = 0.18\mathrm{e}^{-\mathrm{j}2.27}$, $\omega_1 = 5$; $K = 1$, $\mu = 1$, $\sigma^2 = 0.6$, $\omega = \{0.5, 1, 2, 4, 5, 8, 15\}$; $V_\omega^{n,m}(\Pi_1)$ - blue line; $V_\omega^{n,m}(\Pi_2)$ - green line.

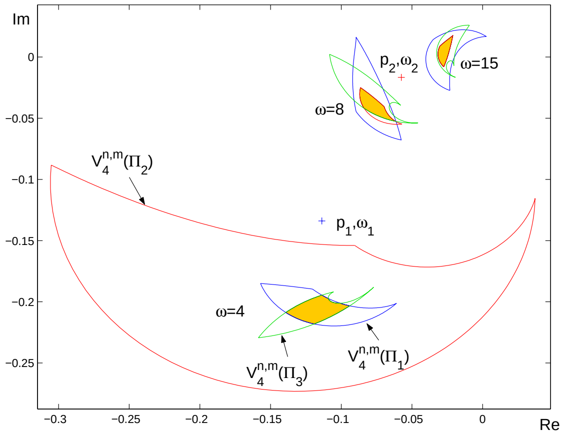

Fig. 4. Computing hard bounds of the value set $\mathcal{V}_\omega^{n,m}(\Pi)$ for the case, $p_1 = 0.18\mathrm{e}^{-\mathrm{j}2.27}$, $\omega_1 = 5$; $p_2 = 0.05\mathrm{e}^{-\mathrm{j}2.86}$, $\omega_2 = 10$; $K = 1$, $\mu = 1$, $\sigma^2 = 0.6$, $\omega = \{4, 8, 15\}$; $V_\omega^{n,m}(\Pi_1)$ - blue line; $V_\omega^{n,m}(\Pi_2)$ - red line; $V_\omega^{n,m}(\Pi_3)$ - green line.

$$\Pi = \{F(\mathrm{j}\omega_1); F(\mathrm{j}\omega_2); F(0), F'(0), F''(0)\} = \{\Pi_1; \Pi_2; \Pi_3\}, \qquad (13)$$

In this case, the necessary and sufficient condition of model set emptiness cannot be formed easily using intersections of value sets as in two cases mentioned above. Obviously, the model set is empty, if $F(j\omega_1) \notin V_{\omega_1}^{n,m}(\Pi_2) \cap V_{\omega_1}^{n,m}(\Pi_3)$ or $F(j\omega_2) \notin V_{\omega_2}^{n,m}(\Pi_1) \cap V_{\omega_2}^{n,m}(\Pi_3)$ An example is shown in Fig. 4. The model set uncertainty for this combination of experimental data is not reduced significantly comparing to previous case (12). Additionally two disadvantages appear. Firstly, it is almost impossible to measure consistent experimental data creating non-empty model set on a real object. This is caused mainly by some noises in measurement and nonlinearities. The second disadvantage is that even if one obtains experimental data creating non-empty intersections of all corresponding value sets, the unfalsified transfer function satisfying all interpolating conditions

Π_1, Π_2 and Π_3 need not to exist. The same disadvantages hold true also for all more complicated sets of interpolating conditions.

4. ROBUST CONTROLLER DESIGN

In this Section, the investigated results will be applied to robust controller design in frequency domain. It is well known that typical design specifications (e.g. gain and phase margins, sensitivity functions constraints, proper bandwidth, disturbance dumping) can be transformed to the graphical limitations on the Nyquist plot in the frequency domain. The design specifications must be fulfilled for all processes contained in the model set. It can be ensured using value sets and their intersections. Consider the controller described by transfer function $C(s)$. Then the hard bounds of closed loop value sets $\mathcal{W}_\omega^{n,m}(\Pi)$ can be computed for each frequency $\omega > 0$ as $\mathcal{W}_\omega^{n,m}(\Pi) = (V_\omega^{n,m}(\Pi_1) \cap \ldots \cap V_\omega^{n,m}(\Pi_k)) {*} C(\mathrm{j}\omega)$. The requirements on the Nyquist plot are fulfilled if they are satisfied for $\mathcal{W}_\omega^{n,m}(\Pi)$ for each $\omega > 0$. Fortunately, only a few closed loop value set on critical frequencies have to be checked because of *a priori* restriction on the transfer function form.

Further, the example will be given for a simple PID controller for the experimental data presented in Fig.4. The PID controller is described by transfer function

$$C(s) = K\left(1 + \frac{1}{T_I s} + \frac{T_D s}{\frac{T_D}{N} s + 1}\right), \tag{14}$$

where the derivative filter parameter is fixed to $N = 10$. The following design specifications have to be fulfilled:

$$\begin{aligned} P_m &> 60^o, \\ M_S &= 1.6, \\ M_T &= 1.6, \end{aligned} \tag{15}$$

where Pm is the phase margin, M_S is the upper limit of the sensitivity function and M_T is the upper limit of the complementary sensitivity function.

Firstly, only one frequency response sample p_1 is considered. The model is very uncertain, the closed loop value sets are large and the controller is very conservative. The controller parameters for this case are $K = 3.31,\ T_I = 0.72,\ T_D = 0.18$.

If only the moments of the impulse response $K = 1,\ \mu = 1, \sigma^2 = 0.6$ are available, the model set uncertainty at critical frequencies is also quite big. The following controller parameters satisfying design specifications (16) for closed loop value sets were obtained: $K = 3.54,\ T_I = 0.55,\ T_D = 0.13$.

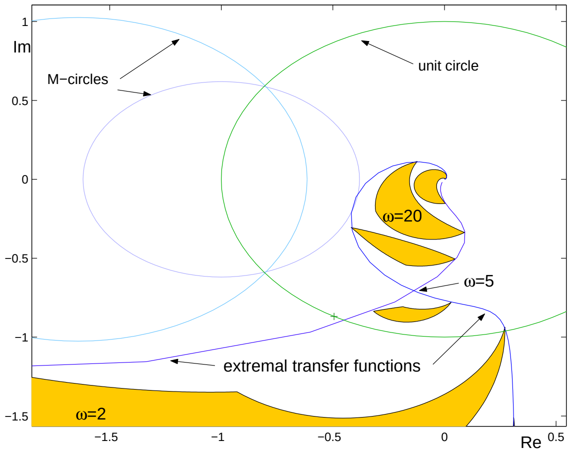

Fig. 5. Robust PID controller satisfying design specifications 16, one sample of the frequency response

Now, let us consider the whole set of interpolating conditions presented in Fig. 4. The intersections of corresponding closed loop value sets create the upper hard bounds of closed loop value sets in the Nyquist plot plane. The intersection is smaller thus the model uncertainty is reduced significantly also in Nyquist plot plane. Therefore, the arising controller satisfying the same design specifications is much less conservative with parameters $K = 6.28,\ T_I = 0.52,\ T_D = 0.13$. For example the static gain is in this case almost two times greater than in previous two cases.

Design method

The robustness regions method (Schlegel et al. (2003)) was used for PID controller design. If the hard bounds are determined as intersections of corresponding value sets, one problem arises comparing to simple cases presented in previous papers. The parameterization of the boundary of the value sets intersections is dependent on frequency ω. Therefore, it is impossible to choose e. g. two extremal processes creating the boundary of the band of all frequency responses and apply any robust design method to these extremal transfer functions. The controller design should by done iteratively. One has to find the 'central' transfer function passing through all intersections in frequency domain and find the controller parameters for this process firstly. After that, the controller parameters have to be changed manually to fulfil all design specifications for all $\mathcal{W}_\omega^{n,m}(\Pi)$, $\omega > 0$. It is necessary to use some interactive tool (for example free accessible 'Fractional PID laboratory' Java applet), otherwise the design procedure is very time consuming.

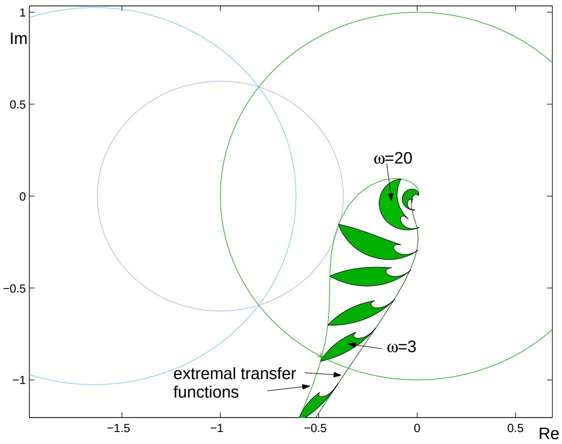

Fig. 6. Robust PID controller satisfying design specifications 16, moments of the impulse response

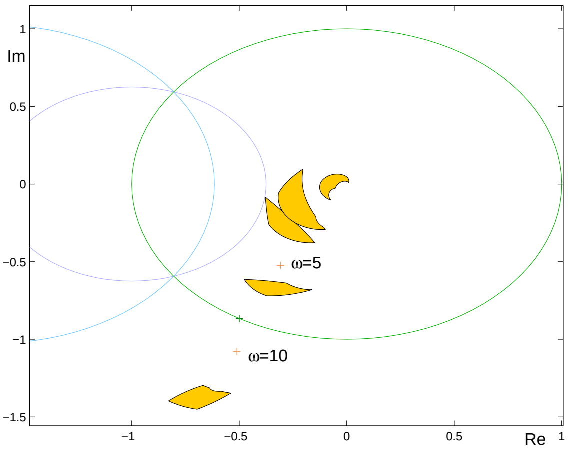

Fig. 7. Robust PID controller satisfying design specifications 16, combination of characteric numbers

5. CONCLUSIONS

The paper extends the authors' previous results where the class of all lag/dead time process models is considered with two special types of general frequency response interpolating conditions - one sample of the frequency response and first three moments of the impulse response. In this paper, the simultaneous knowledge of above interpolating conditions is assumed. Using previous results, the upper hard bounds of value set boundary is computed as an intersection of corresponding value sets. It is shown, that combining experimental data leads to the significant reduction of a model uncertainty in the frequency domain. It is shown that one sample of the frequency response and moments of the impulse response is the perspective combination of characteristic numbers. Additionally, the investigated results are applied to the robust PID controller design. Note, that all figures in this paper were obtained using 'Fractional PID laboratory' Java applet free accessible on www.PIDlab.com. Further, these results could be extended to fractional PID controller design. The problem is also the simulation in time domain for the fractional order pole transfer functions.

References

A.C. Charef, H.H. Sun, Y.Y. Tsao, and B. Onaral. Fractal system as represented by singularity function. *IEEE Transactions on Automatic Control*, 37:1465–1470, 1992.

I. Podlubny. Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers. *IEEE Transactions on Automatic Control*, 44:208–214, 1999.

K.J. Åström and T. Hägglund. The future of PID control. *Control Engineering Practise*, 9:1163–1175, 2001.

M. Schlegel. Exact revision of the ziegler-nichols frequency response method. *in: Proc. of IASTED Int. Conf. on control and applications, Cancun, Mexico*, 2002.

M. Schlegel, P. Balda, and M. Štětina. Robust PID autotuner- method of moments. *Automatizace*, 46(4), 2000.

M. Schlegel and M. Čech. Fractal system identification for robust control: method of moments. *in: ICCC 2004 Summaries Volume*, 2004.

M. Schlegel and M. Čech. Computing value sets from one of the frequency response with applications. *in: Proceedings of IFAC 2005*, 2005.

M. Schlegel, M. Čech, and J. Mertl. Generalized robustness regions for PID controllers. *in: Process Control 2003 Summaries Volume*, 2003.