DESIGN OF $\mathcal{H}_\infty$ PI CONTROLLERS BY ROBUSTNESS REGIONS METHOD

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Abstract: This paper presents the complete analytical solution of the mixed sensitivity design problem for general SISO systems and the PI controller. The generalized robustness regions method is employed to obtain robustness regions in the parameter plane $k - k_i$ for given γ. The value of γ when the admissible region turns into the point corresponds to the optimal PI controller. Moreover, the presented method fully describes the set of all PI controllers which guarantee the given value of the mixed sensitivity criterion γ. The presented solution can also be extended to the PID controller with fixed ratio T_d/T_i or to another fixed structure controller with two design parameters.

Keywords: robust control, $\mathcal{H}_\infty$design, PI control, robustness regions

1 INTRODUCTION

The mixed sensitivity controller design problem for single input single output (SISO) systems deals with sensitivity and complementary sensitivity functions shaping in the frequency domain while the robust stability condition is satisfied [Kwakernaak, 1993].

Let the nominal process model transfer function G_0 is stable and let the nominal open loop transfer function $L_0(j\omega) = C(j\omega)G_0(j\omega)$, where C is a controller transfer function, does not encircle the point -1 in the complex plain. The question is how large perturbation can be involved in the open loop transfer function $L(j\omega)$ to preserve the closed loop stable. The Nyquist stability criterion says that the distance between $L(j\omega)$ and $L_0(j\omega)$ has to be smaller than the distance between $L_0(j\omega)$ and the point -1 for all corresponding frequencies ω. In other words

$$|L(j\omega) - L_0(j\omega)| < |L_0(j\omega) + 1|, \qquad \forall\omega \in \mathcal{R}.$$

It is equivalent with

$$\frac{|L(j\omega) - L_0(j\omega)|}{|L_0(j\omega)|}\frac{|L_0(j\omega)|}{|L_0(j\omega) + 1|} = \frac{|L(j\omega) - L_0(j\omega)|}{|L_0(j\omega)|}|T_0(j\omega)| < 1, \qquad \forall\omega \in \mathcal{R},$$

where T_0 is the nominal closed loop. Now, the relative error of the open loop can be bounded by a frequency dependent function v

$$\frac{|L(j\omega) - L_0(j\omega)|}{|L_0(j\omega)|} \leq v(j\omega), \qquad \forall\omega \in \mathcal{R}. \tag{1}$$

Then if the following inequality holds

$$|v(j\omega)T_0(j\omega)| < 1, \qquad \forall\omega \in \mathcal{R}, \tag{2}$$

the closed loop is stable for all perturbations satisfying (1). Furthermore, it could be proven that (2) is not only sufficient but also necessary condition of closed loop stability for all perturbations bounded by expression (1).

The open loop transfer function satisfying (1) can by rewritten as $L = L_0(1 + \delta v)$ where δ is a frequency dependent function satisfying $|\delta(j\omega)| \leq 1$. This situation is depicted in fig. 1a and it is a special case of the standard perturbation model depicted in fig. 1b. Now, a ∞-norm of SISO system $F(s)$ is defined: $\|F\|_\infty = \sup_{\omega \in \mathcal{R}} |F(j\omega)|$. Denote $H = vT$ and let γ is a positive real number. From (2) it follows that

$$\|H\|_\infty < \gamma, \tag{3}$$

is necessary and sufficient condition for closed loop stability for all perturbations δ satisfying

$$\|\delta\|_\infty \leq \frac{1}{\gamma}. \tag{4}$$

This statement is also valid for multiple input – multiple output processes (see Small Gain Theorem for instance in [Zhou *et al.*, 1996]). The ∞-norm of the transfer functions matrix F is defined as $\|F\|_\infty = \sup_{\omega \in \mathcal{R}} \|F(j\omega)\|_2$ where $\|F\|_2$ denotes the 2-norm of the matrix F. For a constant complex-valued matrix F the norm $\|F\|_2 = \max_i \sigma_i(F)$ where $\sigma_i(F)$ is the i-th singular value of the matrix F. The ∞-norm, $\|F\|_\infty$, of stable process with a transfer functions matrix F can be found by computing the highest singular values of the matrix $F(j\omega)$ for all frequencies and then by choosing a maximum among them. For more details see [Kwakernaak, 1993].

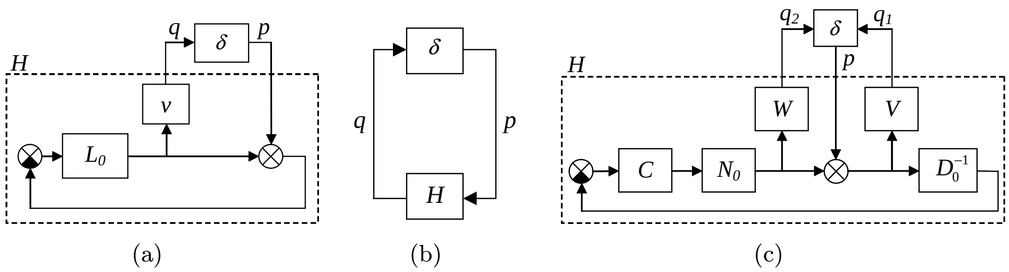

Figure 1 - (a) A feedback loop with perturbations, (b) A standard model with perturbations, (c) The mixed sensitivity problem.

The mixed sensitivity problem [Kwakernaak, 1985] can be transformed to form of a standard model H with perturbations δ, see fig. 1b. Now, let G is a process with perturbations in both its numerator and denominator in the form

$$G = \frac{N}{D} = \frac{N_0(1 + \delta_W W)}{D_0(1 + \delta_V V)}, \tag{5}$$

where $G_0 = N_0/D_0$ denotes the transfer function of its nominal model, W and V are frequency dependent functions with the meaning of maximal allowed relative errors of the numerator and the denominator respectively and

$$\delta_W = \frac{\frac{N - N_0}{N_0}}{W}, \qquad \delta_V = \frac{\frac{D - D_0}{D_0}}{V} \tag{6}$$

are frequency dependent functions with their amplitudes lesser than one. A control loop containing the process (5) can be represented by a scheme depicted in fig. 1c where the block H from fig. 1b is emphasized by a dashed line. The whole system can be described by equations

$$[q_1 \ q_2]^T = Hp, \qquad p = \delta[q_1 \ q_2]^T,$$

where

$$H = \begin{bmatrix} \frac{V}{1+CG_0} \\ -\frac{WCG_0}{1+CG_0} \end{bmatrix} = \begin{bmatrix} VS_0 \\ -WT_0 \end{bmatrix}, \qquad \delta = [-\delta_V \ \delta_W],$$

S_0 and T_0 denote the nominal sensitive and the nominal complementary sensitive functions respectively defined by following expressions:

$$T(s) \triangleq \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{L(s)}{1+L(s)}, \ S(s) \triangleq \frac{1}{1+G(s)C(s)} = \frac{1}{1+L(s)}. \tag{7}$$

Using (3), (4) and (6) the following lemma can be formed [Kwakernaak, 1993].

Lemma 1 *The closed loop containing the process (5) is stable for all perturbations satisfying*

$$\left| \frac{\frac{N(j\omega)-N_0(j\omega)}{N_0(j\omega)}}{W(j\omega)} \right|^2 + \left| \frac{\frac{D(j\omega)-D_0(j\omega)}{D_0(j\omega)}}{V(j\omega)} \right|^2 \leq \frac{1}{\gamma}, \qquad \forall \omega \in \mathcal{R} \tag{8}$$

iff

$$|V(j\omega)S_0(j\omega)|^2 + |W(j\omega)T_0(j\omega)|^2 < \gamma, \qquad \forall \omega \in \mathcal{R}, \tag{9}$$

where γ is a positive real number.

To design a controller for a SISO process using the mixed sensitivity method, suitable weighting functions $V(j\omega)$, $W(j\omega)$ have to be chosen first and then a stabilizing controller minimizing the criterion

$$\gamma = \sup_{\omega \in \mathcal{R}} |V(j\omega)S_0(j\omega)|^2 + |W(j\omega)T_0(j\omega)|^2 \tag{10}$$

is computed. Usually, the amplitude of the sensitivity function should be small at low frequencies and the amplitude of the complementary sensitive function should be small at high frequencies; it can be achieved by choosing high values values of V at low frequencies and high values of W at high frequencies. Because functions W and V are measures of relative perturbations of the numerator and the denominator respectively high frequency perturbations should be modelled as numerator perturbations and low frequency perturbations as denominator perturbations.

2 GEOMETRICAL INTERPRETATION OF MIXED SENSITIVITY PROBLEM

In this section an original interpretation of mixed sensitivity criterion by a form of 'forbidden' and 'allowed' circles for separate points of Nyquist curve is presented. The mixed sensitivity criterion (9) can be rewritten as

$$\frac{|V(j\omega)|^2}{|1+L_0(j\omega)|^2} + \frac{|W(j\omega)|^2|L_0(j\omega)|^2}{|1+L_0(j\omega)|^2} < \gamma, \qquad \forall \omega \geq 0$$

where $L_0(j\omega) = C(j\omega)G_0(j\omega)$ is the nominal open loop. This expression can be without loss of generality rewritten to the form

$$\frac{p^2(\omega)}{[1+u(\omega)]^2 + v^2(\omega)} + \frac{g^2(\omega)[u^2(\omega)+v^2(\omega)]}{[1+u(\omega)]^2 + v^2(\omega)} < 1, \qquad \forall \omega \geq 0, \tag{11}$$

where

$$p(\omega) \triangleq \frac{|V(j\omega)|}{\sqrt{\gamma}}, \qquad g(\omega) \triangleq \frac{|W(j\omega)|}{\sqrt{\gamma}}, \qquad L_0(j\omega) \triangleq u(\omega) + jv(\omega). \tag{12}$$

Hereafter the functions of variable ω will be used without the symbol (ω). Thus (11) can be transformed to the form

$$(g^2-1)\left[\left(u - \frac{1}{g^2-1}\right)^2 + v^2\right] < \frac{1-(g^2-1)(p^2-1)}{g^2-1}, \qquad \forall \omega \geq 0. \tag{13}$$

If the equality of left and right hand sides of (13) is supposed, we obtain for a fixed ω the equation of a circle with radius r and center c for which it holds

$$c = \frac{1}{g^2-1}, \tag{14}$$

$$r^2 = \frac{1-(g^2-1)(p^2-1)}{(g^2-1)^2}. \tag{15}$$

Now, the question is where $L_0(j\omega)$ satisfying the inequality (13) must be located in the complex plane depending on value of g.

- For $g \in \langle 0, 1)$ it holds: $c \in \langle -1, -\infty)$ and the right hand side of equation (15) is always positive. Because of $g^2-1 < 1$, the direction of the inequality (13) divided by this expression will change. Thus the point $L_0(j\omega)$ has to lie outside of a circle with the center c and the radius r in the complex plain.
- For $g = 1$ the circle become a line parallel with the imaginary axis crossing the real axis in a point $0{,}5p^2 - 0{,}5$. The point $L_0(j\omega)$ has to lie on the right hand side of the line in the complex plain.
- For $g \in (1, \infty)$ it holds $c \in (+\infty, 0)$. The right hand side of the equation (15) is negative for $p > g/\sqrt{g^2-1}$. But in such case the inequality (13) cannot be satisfied and so the point $L_0(j\omega)$ satisfying (13) has to lie in a circle with the center c and the diameter r in the complex plain.

3 ROBUSTNESS REGIONS FOR MIXED SENSITIVITY PROBLEM AND PI CONTROLLER

Designing a controller using mixed sensitivity method means to find such controller parameters that minimize the value γ of criterion (10) for given weighting functions $V(j\omega)$ and $W(j\omega)$ while the nominal stability condition is satisfied. In the case of PI controller with transfer function

$$C(s) = k\left(1 + \frac{1}{T_i s}\right) \triangleq k + \frac{k_i}{s} = k + \frac{kd}{s}. \tag{16}$$

it is possible to find a region in the parameter plane k–k_i [Schlegel, 2003] for sufficiently high γ that each point of the region has such property that control loop satisfies inequality (9). Then, the value of γ can be decreased as long as the region become a point in the parametric plane k–k_i.

Let $V(j\omega)$, $W(j\omega)$ are given smooth weighting functions, $G_0(j\omega)$ is a given nominal process model and γ is a given positive real number. We are looking for such region in the parametric plane k–k_i that for each point of the region the inequality (11) is satisfied. The equality in (11) has to hold for each point of the boundary of the region at some fixed frequency ω_x. Thus for boundary point it holds

$$(u+1)^2 + v^2 - p^2 - g^2(u^2+v^2) = 0. \tag{17}$$

Because at the same ω_x the left hand side of (11) has the extrem and $V(j\omega)$, $W(j\omega)$ and $L(j\omega)$ are smooth the following equation must also hold

$$\begin{array}{l} \frac{d}{d\omega}\left\{\frac{p^2}{(1+u)^2+v^2} + \frac{g^2(u^2+v^2)}{(1+u)^2+v^2}\right\}\Big|_{\omega=\omega_x} = \\ 2\frac{\left[pp_1+gg_1(u^2+v^2)+g^2(uu_1+vv_1)\right]\left[(u+1)^2+v^2\right]-\left[p^2+g^2(u^2+v^2)\right]\left[uu_1+u_1+vv_1\right]}{\left[(u+1)^2+v^2\right]^2}\Bigg|_{\omega=\omega_x} = 0, \end{array}$$

where

$$\frac{d}{d\omega}p(\omega) \triangleq p_1, \qquad \frac{d}{d\omega}g(\omega) \triangleq g_1, \qquad \frac{d}{d\omega}L(j\omega) \triangleq u_1 + jv_1.$$

Now, using the fact that the left hand side of the inequality (11) is equal to one at the frequency ω_x we obtain

$$pp_1 + gu^2g_1 + g^2uu_1 + gv^2g_1 + g^2vv_1 - u_1 - uu_1 - vv_1 = 0, \tag{18}$$

Replacing variables u, v, u_1, v_1 according to expression

$$L(j\omega) = C(j\omega)G(j\omega) = \left(k + \frac{kd}{j\omega}\right)(a+jb)$$

in equations (17) and (18) produces an equation system

$$\begin{array}{l} \left(ak + \frac{bkd}{\omega} + 1\right)^2 + \left(-\frac{akd}{\omega} + bk\right)^2 - p^2 - \\ g^2\left[\left(ak + \frac{bkd}{\omega}\right)^2 + \left(-\frac{akd}{\omega} + bk\right)^2\right] = 0, \end{array} \tag{19}$$

$$\begin{array}{l} \Big\{pp_1 + gg_1\left[\left(ak + \frac{bkd}{\omega}\right)^2 + \left(-\frac{akd}{\omega} + bk\right)^2\right] + \\ g^2\left[\left(ak + \frac{bkd}{\omega}\right)\left(a_1k + \frac{b_1kd}{\omega} - \frac{bkd}{\omega^2}\right) + \left(-\frac{akd}{\omega} + bk\right)\left(\frac{-a_1kd}{\omega} + \frac{akd}{\omega^2} + b_1k\right)\right]\Big\} \\ \left\{\left(ak + \frac{bkd}{\omega} + 1\right)^2 + \left(-\frac{akd}{\omega} + bk\right)^2\right\} - \\ \left\{p^2 + g^2\left[\left(ak + \frac{bkd}{\omega}\right)^2 + \left(-\frac{akd}{\omega} + bk\right)^2\right]\right\}\Big\{\left(ak + \frac{bkd}{\omega} + 1\right) \\ \left(a_1k + \frac{b_1kd}{\omega} - \frac{bkd}{\omega^2}\right) + \left(-\frac{akd}{\omega} + bk\right)\left(-\frac{a_1kd}{\omega} + \frac{akd}{\omega^2} + b_1k\right)\Big\} = 0. \end{array} \tag{20}$$

The problem of solving these equations for indeterminate parameters k and d can be transformed to a problem of finding roots of second and fourth order polynomials, see [Schwarz, 1968]. The solution is given in Appendix A. Let ρ_l, $l = 1, 2, \ldots, m$ denote real roots of these polynomials then a pair $(k \triangleq k(\rho_l), k_i \triangleq k(\rho_l)d(\rho_l))$, $l = 1, 2, \ldots, m$ denote parametric curves with parameter ω that represent boundaries of regions in the parameter plane k–k_i. Then, if we move a point (k, k_i) from one region to another, there exists at least one frequency ω for which (17) and (18) hold when we are crossing the region boundary.

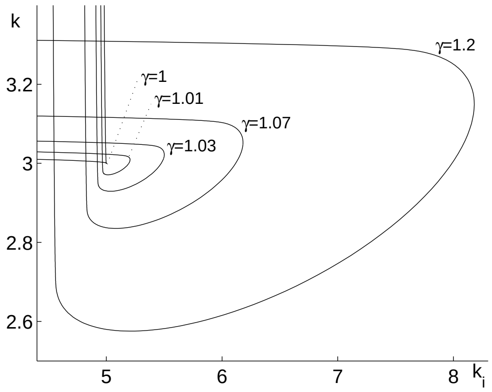

Figure 2 - Robustness regions for mixed sensitivity criterion.

Example 1 *Let us design the PI controller for a first order nominal process model*

$$G(s) = \frac{1}{\tau s + 1}$$

for the weighting functions

$$V(j\omega) = \frac{\omega_B}{j\omega}, \quad W(j\omega) = \frac{j\omega}{\omega_B}$$

where $\tau = 0.6$, $\omega_B = 5$. *Figure 2 shows robustness regions in the parameter plane* $k - k_i$ *for different values of* γ. *For* $\gamma = 1$ *the region turns into the optimal point* $k = 3$ *and* $k_i = 5$.

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APPENDIX A

Transformation of Problem of Finding a Solution to Equation System (19) and (20) to Problem of Finding Roots of Polynomials

The problem of finding a solution to the equation system (19) and (20) with unknown variables k and d can be transformed to the two separate problems of finding roots of second and fourth order polynomials for a constant frequency ω. We are interested in only real roots of these polynomials because of their physical meaning. The transformation problem was solved using symbolic math computation in Maple, problem of finding roots of polynomials and enumerating searched unknowns was numerically solved in Matlab.

I. Second order polynomial. First (maximally two) solutions can be obtained by finding roots of polynomial

$$c_2\rho^2 + c_1\rho + c_0 = 0,$$

where coefficients c_i are defined by following expressions

$$c_2 = 4, \qquad c_1 = 4g^2a - 4ap^2, \qquad c_0 = -2g^2a^2p^2 + a^2p^4 + p^4b^2 + g^4a^2 + 2g^2p^2b^2 + g^4b^2.$$

Let ρ subsequently pass over all real roots, the searched unknowns k and d are defined by following expressions:

$k = \rho/g^2/(a^2 + b^2)$,

$$d = -1/2(-b^6p^6 + (6\rho p^4b^4 + (-p^6b^4 + (4\rho p^4b^2 + (p^6b^2 + (-2\rho p^4 + p^6a)a)a)a)a)a + ((4\rho\omega p^2a_1 + (2p^4\omega b_1 + (-3 + 3p^2)p^4b)b)b^4 + ((-16\rho\omega p^2b_1 + (6p^4\omega a_1 + (12 - 18p^2)p^2\rho)b)b^3 + ((-24\rho\omega p^2a_1 + (-4p^4\omega b_1 + (3 + 3p^2)p^4b)b)b^2 + ((16\rho\omega p^2b_1 + (4p^4\omega a_1 + (8 - 12p^2)p^2\rho)b)b + (4\rho\omega p^2a_1 + (-6p^4\omega b_1 + (7 - 3p^2)p^4b)b + (-2p^4\omega a_1 + (-4 + 6p^2)p^2\rho + (1 - 3p^2)p^4a)a)a)a)a)a + (-4b^6p^4\omega g_1 + (24\rho p^2b^4\omega g_1 + (-4\omega p^4b^4g_1 + (16\rho p^2b^2\omega g_1 + (4p^4b^2\omega g_1 + (-8\rho p^2\omega g_1 + 4p^4\omega g_1a)a)a)a)a)a + (((-4 - 8p^2)a_1\omega\rho + ((1 + (3 - 3p^2)p^2)p^2 - 4\omega p^3p_1)b^2)b^4 + (((8 + 24p^2)b_1\omega\rho + ((8 - 8p^2)p^2a_1\omega + (-2 + (-12 + 18p^2)p^2 + 8\omega pp_1)\rho)b)b^3 + ((40\rho\omega p^2a_1 + ((-24 + 16p^2)p^2b_1\omega + ((17 + (-21 - 3p^2)p^2)p^2 - 12\omega p^3p_1)b)b)b^2 + (((-40p^2 + 8)b_1\omega\rho + (-24\omega p^2a_1 + (-4 + 12p^4 + 16\omega pp_1)\rho)b)b + ((4 - 16p^2)a_1\omega\rho + ((8 + 16p^2)p^2b_1\omega + ((15 + (-27 + 3p^2)p^2)p^2 - 12\omega p^3p_1)b)b + (8p^4\omega a_1 + (-2 + (12 - 6p^2)p^2 + 8\omega pp_1)\rho + ((-1 + (-3 + 3p^2)p^2)p^2 - 4\omega p^3p_1)a)a)a)a)a + (-4p^2b^6\omega g_1 + (-16\rho p^2b^4\omega g_1 + ((20g_1 - 8p^2g_1)p^2\omega b^4 + ((20g_1 - 16p^2g_1)p^2\omega b^2 + (16\rho p^2\omega g_1 + (-4g_1 - 8p^2g_1)p^2\omega a)a)a^2)a)a + (((4p^2 + 8)a_1\omega\rho + ((-2 + (4 - 6p^2)p^2)b_1\omega + (3 + (-7 + (3 + p^2)p^2)p^2 + (4p_1 + 8p_1p^2)p\omega)b)b)b^4 + ((-32\rho\omega b_1 + ((-6 + (-4 - 2p^2)p^2)a_1\omega + (14 + (-12 - 6p^2)p^2 - 16\omega pp_1)\rho)b)b^3 + (((-8p^2 - 32)a_1\omega\rho + ((4 + (40 - 20p^2)p^2)b_1\omega + (5 + (-35 + (33 + p^2)p^2)p^2 + (12p_1 + 24p_1p^2)p\omega)b)b)b^2 + ((32\rho\omega p^2b_1 + ((-4 + (56 - 12p^2)p^2)a_1\omega + (20 + (-24 - 4p^2)p^2 - 32\omega pp_1)\rho)b)b + ((20p^2 - 8)a_1\omega\rho + ((6 + (-28 - 14p^2)p^2)b_1\omega + (1 + (-25 + (33 - p^2)p^2)p^2 + (12p_1 + 24p_1p^2)p\omega)b)b + ((2 + (-4 - 10p^2)p^2)a_1\omega + (6 + (-12 + 2p^2)p^2 - 16\omega pp_1)\rho + (-1 + (3 + (3 - p^2)p^2)p^2 + (4p_1 + 8p_1p^2)p\omega)a)a)a)a)a + ((8g_1 + (-8g_1 + 4p^2g_1)p^2)\omega b^6 + ((-8p^2g_1 + 32g_1)\omega\rho b^4 + ((16g_1 + (-40g_1 + 12p^2g_1)p^2)\omega b^4 + ((-16p^2g_1 + 32g_1)\omega\rho b^2 + ((8g_1 + (-24g_1 + 12p^2g_1)p^2)\omega b^2 + (-8\rho p^2\omega g_1 + (8g_1 + 4p^2g_1)p^2\omega a)a)a)a)a + ((-4\rho\omega a_1 + ((-4 + 4p^4)b_1\omega + (1 + (3 - 3p^2)p^2 + (-8p_1 - 4p_1p^2)p\omega)b)b)b^4 + (((-8p^2 + 24)b_1\omega\rho + ((-4 + 4p^4)a_1\omega + (12p^2 + 8\omega pp_1 - 6)\rho)b)b^3 + (((-8p^2 + 32)a_1\omega\rho + ((-8 + (-8 + 8p^2)p^2)b_1\omega + (5 + (3 - 15p^2)p^2 + (-24p_1 - 12p_1p^2)p\omega)b)b)b^2 + (((-8p^2 - 8)b_1\omega\rho + ((-8 + (-24 + 8p^2)p^2)a_1\omega + (16\omega pp_1 - 12 + 16p^2)\rho)b)b + ((4 - 8p^2)a_1\omega\rho + ((-4 + (24 + 4p^2)p^2)b_1\omega + (7 + (-3 - 13p^2)p^2 + (-24p_1 - 12p_1p^2)p\omega)b)b + ((-4 + (4p^2 + 8)p^2)a_1\omega + (4p^2 + 8\omega pp_1 - 6)\rho + (3 + (-3 - p^2)p^2 + (-8p_1 - 4p_1p^2)p\omega)a)a)a)a)a + ((8g_1 - 4p^2g_1)\omega b^6 + ((-12p^2g_1 + 16g_1)\omega b^4 + ((-12p^2g_1 + 8g_1)\omega b^2 - 4p^2\omega g_1a^2)a^2)a^2 + (((-4p^2 + 6)b_1\omega + (4\omega pp_1 + 3p^2 - 3)b)b^5 + (((10 - 4p^2)a_1\omega - 6\rho)b^4 + (((4 - 8p^2)b_1\omega + (15p^2 - 9 + 12\omega pp_1)b)b^3 + (((-8p^2 + 12)a_1\omega - 4\rho)b^2 + (((-4p^2 - 2)b_1\omega + (13p^2 + 12\omega pp_1 - 9)b)b + ((-4p^2 + 2)a_1\omega + 2\rho + (-3 + p^2 + 4\omega pp_1)a)a)a)a)a)a + (-b^6 + (-b^4 + (a^2 + b^2)a^2)a^2)g^2)g)g)g)g)g)g)g)g^2)\omega/(\rho p^4b^5 + (p^6b^5 + (-2\rho p^4b^3 + (2p^6b^3 + (-3\rho p^4b + p^6ba)a)a)a + ((-2\rho\omega p^2b_1 + (p^4\omega a_1 + (4 - 3p^2)p^2\rho)b)b^4 + ((-8\rho\omega p^2a_1 + (-3p^4\omega b_1 + (4 - 3p^2)p^4b)b)b^3 + ((12\rho\omega p^2b_1 + (6\rho p^4 - 2p^4\omega a_1)b)b^2 + ((8\rho\omega p^2a_1 + (-2p^4\omega b_1 + (4 - 6p^2)p^4b)b)b + (-2\rho\omega p^2b_1 + (-3p^4\omega a_1 + (-4 + 9p^2)p^2\rho)b + (-3p^6b + p^4\omega b_1)a)a)a)a)a + (4\rho p^2\omega b^5g_1 + (4p^4\omega b^5g_1 + (-8\rho p^2\omega b^3g_1 + (8p^4\omega b^3g_1 + (-12\rho p^2\omega bg_1 + 4p^4\omega bg_1a)a)a)a)a + ((-2\rho\omega b_1 + ((2 - 2p^2)p^2a_1\omega + (3 + (-6 + 3p^2)p^2 + 4\omega pp_1)\rho)b)b^4 + (((-4 + 12p^2)a_1\omega\rho + ((-6 + 6p^2)p^2b_1\omega + (5 + (-9 + 3p^2)p^2)p^2b)b)b^3 + ((-28\rho\omega p^2b_1 + ((-8 + 4p^2)p^2a_1\omega + (6 + 8\omega pp_1 - 6p^4)\rho)b)b^2 + (((-4 - 20p^2)a_1\omega\rho + ((4p^2 + 8)p^2b_1\omega + (2 + (-6 + 6p^2)p^2)p^2b)b)b + ((4p^2 + 2)b_1\omega\rho + ((6 + 6p^2)p^2a_1\omega + (3 + (6 - 9p^2)p^2 + 4\omega pp_1)\rho)b + ((-2 - 2p^2)p^2b_1\omega + (-3 + (3 + 3p^2)p^2)p^2b)a)a)a)a + (8\rho\omega b^5g_1 + ((8g_1 - 4p^2g_1)p^2\omega b^5 + ((16p^2g_1 + 16g_1)\omega\rho b^3 + (-8p^4\omega b^3g_1 + ((16p^2g_1 + 8g_1)\omega\rho b + (-8g_1 - 4p^2g_1)p^2\omega ba)a)a)a)a + (((-4 + 6p^2)b_1\omega\rho + ((1 + (-4 + p^2)p^2)a_1\omega + (1 - p^4 - 8\omega pp_1)\rho)b)b^4 + ((-8\rho\omega a_1 + ((-3 + (12 - 3p^2)p^2)b_1\omega + (2 + (-9 + (6 - p^2)p^2)p^2)b)b)b^3 + (((20p^2 + 8)b_1\omega\rho + ((-2 + (16 - 2p^2)p^2)a_1\omega + (-16\omega pp_1 + 2p^4 - 2)\rho)b)b^2 + (((8 + 16p^2)a_1\omega\rho + ((-2 + (-16 - 2p^2)p^2)b_1\omega + (4 + (-6 - 2p^4)p^2)b)b)b + ((-4 - 2p^2)b_1\omega\rho + ((-3 + (-12 - 3p^2)p^2)a_1\omega + (-8\omega pp_1 - 3 + 3p^4)\rho)b + ((1 + (4 + p^2)p^2)b_1\omega + (2 + (3 + (-6 - p^2)p^2)p^2)b)a)a)a)a + ((8g_1 - 4p^2g_1)\omega\rho b^5 + ((-8p^2g_1 + 4g_1)\omega b^5 + (-8\rho p^2\omega b^3g_1 + (8\omega b^3g_1 + ((-8g_1 - 4p^2g_1)\omega\rho b + (8p^2g_1 + 4g_1)\omega ba)a)a)a)a + (((-4p^2 + 6)b_1\omega\rho + ((-2 + 2p^2)a_1\omega + (4\omega pp_1 + 2p^2 - 3)\rho)b)b^4 + (((-4p^2 + 12)a_1\omega\rho + ((-6p^2 + 6)b_1\omega + (-3 + (3 - p^2)p^2)b)b)b^3 + (((-4p^2 - 8)b_1\omega\rho + ((4 - 8p^2)a_1\omega + (8\omega pp_1 - 6)\rho)b)b^2 + (((-4 - 4p^2)a_1\omega\rho + ((8p^2 + 4)b_1\omega + (-6 + (6 + 2p^2)p^2)b)b)b + (2\rho\omega b_1 + ((6 + 6p^2)a_1\omega + (4\omega pp_1 - 3 - 2p^2)\rho)b + ((-2 - 2p^2)b_1\omega + (-3 + (3 + 3p^2)p^2)b)a)a)a)a + ((-4\omega b^5g_1 + (-8\omega b^3g_1 - 4\omega bg_1a^2)a^2)a + ((-\rho + \omega a_1)b^5 + ((-3\omega b_1 + p^2b)b^4 + ((2\rho - 2\omega a_1)b^3 + ((-2\omega b_1 - 2p^2b)b^2 + ((-3\omega a_1 + 3\rho)b + (-3p^2b + \omega b_1)a)a)a)a)a + (b^5 + (2b^3 + ba^2)a^2)ag^2)g)g)g)g)g)g)g)g^2).$$

II. Fourth order polynomial. Next (maximally four) solutions can be obtained by finding roots of polynomial

$$c_4\rho^4 + c_3\rho^3 + c_2\rho^2 + c_1\rho + c_0 = 0,$$

where coefficients c_i are defined by following expressions

$$c_4 = b^8 + (4b^6 + (6b^4 + (4b^2 + a^2)a^2)a^2)a^2 + (-4b^8 + (-16b^6 + (-24b^4 + (-16b^2 - 4a^2)a^2)a^2)a^2 + (6b^8 + (24b^6 + (36b^4 + (24b^2 + 6a^2)a^2)a^2)a^2 + (-4b^8 + (-16b^6 + (-24b^4 + (-16b^2 - 4a^2)a^2)a^2)a^2 + (b^8 + (4b^6 + (6b^4 + (4b^2 + a^2)a^2)a^2)a^2)g^2)g^2)g^2)g^2,$$

$$c_3 = 2a_1\omega b^6 + ((-4\omega b_1 + 4b)b^5 + (2a_1\omega b^4 + ((-8\omega b_1 + 12b)b^3 + (-2a_1\omega b^2 + ((-4\omega b_1 + 12b)b + (-2\omega a_1 + 4a)a)a)a)a)a)a + ((4b^6\omega g_1 + (12b^4\omega g_1 + (12b^2\omega g_1 + 4\omega g_1 a^2)a^2)a^2)a + (-6a_1\omega b^6 + ((12\omega b_1 - 12b)b^5 + (-6a_1\omega b^4 + ((24\omega b_1 - 36b)b^3 + (6a_1\omega b^2 + ((12\omega b_1 - 36b)b + (6\omega a_1 - 12a)a)a)a)a)a)a + ((-8b^6\omega g_1 + (-24b^4\omega g_1 + (-24b^2\omega g_1 - 8\omega g_1 a^2)a^2)a^2)a + (6a_1\omega b^6 + ((-12\omega b_1 + 12b)b^5 + (6a_1\omega b^4 + ((-24\omega b_1 + 36b)b^3 + (-6a_1\omega b^2 + ((-12\omega b_1 + 36b)b + (-6\omega a_1 + 12a)a)a)a)a)a)a + ((4b^6\omega g_1 + (12b^4\omega g_1 + (12b^2\omega g_1 + 4\omega g_1 a^2)a^2)a^2)a + (-2a_1\omega b^6 + ((4\omega b_1 - 4b)b^5 + (-2a_1\omega b^4 + ((8\omega b_1 - 12b)b^3 + (2a_1\omega b^2 + ((4\omega b_1 - 12b)b + (2\omega a_1 - 4a)a)a)a)a)a)a)g)g)g)g)g)g,$$

$$c_2 = ((a_1^2 + b_1^2)\omega^2 + ((-2 + 2p^2)b_1\omega + (-2\omega pp_1 + 1 - 2p^2)b)b)b^4 + ((2p^2 + 2)a_1\omega b^4 + (((2a_1^2 + 2b_1^2)\omega^2 + ((4p^2 - 12)b_1\omega + (-6p^2 + 8 - 6\omega pp_1)b)b)b^2 + ((4p^2 - 4)a_1\omega b^2 + ((a_1^2 + b_1^2)\omega^2 + ((-10 + 2p^2)b_1\omega + (13 - 6p^2 - 6\omega pp_1)b)b + ((-6 + 2p^2)a_1\omega + (-2\omega pp_1 - 2p^2 + 6)a)a)a)a)a)a + ((-4g_1 b_1\omega^2 + (2g_1 - 2p^2 g_1)\omega b)b^5 + (-4g_1 a_1\omega^2 b^4 + ((-8g_1 b_1\omega^2 + (-6p^2 g_1 + 14g_1)\omega b)b^3 + (-8g_1 a_1\omega^2 b^2 + ((-4g_1 b_1\omega^2 + (-6p^2 g_1 + 22g_1)\omega b)b + (-4g_1 a_1\omega^2 + (10g_1 - 2p^2 g_1)\omega a)a)a)a)a)a + (((-2a_1^2 - 2b_1^2)\omega^2 + ((-6p^2 + 6)b_1\omega + (-4 + 6p^2 + (6pp_1 + 4g_1^2\omega)\omega)b)b)b^4 + ((-6p^2 - 2)a_1\omega b^4 + (((-4a_1^2 - 4b_1^2)\omega^2 + ((28 - 12p^2)b_1\omega + (18p^2 - 22 + (18pp_1 + 12g_1^2\omega)\omega)b)b)b^2 + ((-12p^2 + 12)a_1\omega b^2 + ((-2a_1^2 - 2b_1^2)\omega^2 + ((-6p^2 + 22)b_1\omega + (18p^2 - 32 + (18pp_1 + 12g_1^2\omega)\omega)b)b + ((-6p^2 + 14)a_1\omega + (-14 + 6p^2 + (6pp_1 + 4g_1^2\omega)\omega)a)a)a)a)a)a + ((4g_1 b_1\omega^2 + (4p^2 g_1 - 4g_1)\omega b)b^5 + (4g_1 a_1\omega^2 b^4 + ((8g_1 b_1\omega^2 + (-20g_1 + 12p^2 g_1)\omega b)b^3 + (8g_1 a_1\omega^2 b^2 + ((4g_1 b_1\omega^2 + (-28g_1 + 12p^2 g_1)\omega b)b + (4g_1 a_1\omega^2 + (-12g_1 + 4p^2 g_1)\omega a)a)a)a)a)a + (((a_1^2 + b_1^2)\omega^2 + ((-6 + 6p^2)b_1\omega + (-6\omega pp_1 + 5 - 6p^2)b)b)b^4 + ((6p^2 - 2)a_1\omega b^4 + (((2a_1^2 + 2b_1^2)\omega^2 + ((12p^2 - 20)b_1\omega + (-18p^2 + 20 - 18\omega pp_1)b)b)b^2 + ((12p^2 - 12)a_1\omega b^2 + ((a_1^2 + b_1^2)\omega^2 + ((-14 + 6p^2)b_1\omega + (25 - 18p^2 - 18\omega pp_1)b)b + ((-10 + 6p^2)a_1\omega + (-6\omega pp_1 - 6p^2 + 10)a)a)a)a)a)a + ((2g_1 - 2p^2 g_1)\omega b^6 + ((-6p^2 g_1 + 6g_1)\omega b^4 + ((-6p^2 g_1 + 6g_1)\omega b^2 + (2g_1 - 2p^2 g_1)\omega a^2)a^2)a^2 + (((2 - 2p^2)b_1\omega + (-2 + 2\omega pp_1 + 2p^2)b)b^5 + ((2 - 2p^2)a_1\omega b^4 + (((-4p^2 + 4)b_1\omega + (6p^2 + 6\omega pp_1 - 6)b)b^3 + ((-4p^2 + 4)a_1\omega b^2 + (((2 - 2p^2)b_1\omega + (6p^2 + 6\omega pp_1 - 6)b)b + ((2 - 2p^2)a_1\omega + (-2 + 2\omega pp_1 + 2p^2)a)a)a)a)a)a)g)g)g)g)g)g,$$

$$c_1 = (2b_1 a_1\omega^2 p^2 + (-2p^2 a_1 - 2p_1 a_1 p\omega)\omega b)b^3 + ((((-4p^2 + 2)b_1^2 + (2p^2 + 2)a_1^2)\omega^2 + (((8p^2 - 4)b_1 + 4p_1 b_1 p\omega)\omega + (-4\omega pp_1 + 2 - 4p^2)b)b)b^2 + ((-6b_1 a_1\omega^2 p^2 + (4p^2 - 2)a_1\omega b)b + ((2b_1^2 + (2 - 2p^2)a_1^2)\omega^2 + (((-8 + 8p^2)b_1 + 4p_1 b_1 p\omega)\omega + (-8p^2 + 6 - 8\omega pp_1)b)b + (((-6 + 6p^2)a_1 + 2p_1 a_1 p\omega)\omega + (-4p^2 - 4\omega pp_1 + 4)a)a)a)a)a + ((-2g_1 - 2p^2 g_1)a_1\omega^2 b^4 + (((-4g_1 + 8p^2 g_1)b_1\omega^2 + (-8p^2 g_1 + 4g_1 - 4p_1 g_1 p\omega)\omega b)b^3 + ((-8g_1 + 4p^2 g_1)a_1\omega^2 b^2 + (((-4g_1 + 8p^2 g_1)b_1\omega^2 + (12g_1 - 16p^2 g_1 - 8p_1 g_1 p\omega)\omega b)b + ((6p^2 g_1 - 6g_1)a_1\omega^2 + (-8p^2 g_1 + 8g_1 - 4p_1 g_1 p\omega)\omega a)a)a)a)a + (((-4p^2 + 2)b_1 a_1\omega^2 + ((4p^2 - 2)a_1 + 4p_1 a_1 p\omega)\omega b)b^3 + ((((-6 + 8p^2)b_1^2 - 4a_1^2 p^2)\omega^2 + (((-16p^2 + 12)b_1 - 8p_1 b_1 p\omega)\omega + (-6 + 8p^2 + (8pp_1 + (-4g_1^2 p^2 + 4g_1^2)\omega)\omega)b)b)b^2 + (((12p^2 - 6)b_1 a_1\omega^2 + (-8p^2 + 6)a_1\omega b)b + ((-2b_1^2 + (4p^2 - 4)a_1^2)\omega^2 + (((16 - 16p^2)b_1 - 8p_1 b_1 p\omega)\omega + (16p^2 - 14 + (16pp_1 + (-8g_1^2 p^2 + 8g_1^2)\omega)\omega)b)b + (((-12p^2 + 12)a_1 - 4p_1 a_1 p\omega)\omega + (-8 + 8p^2 + (8pp_1 + (-4g_1^2 p^2 + 4g_1^2)\omega)\omega)a)a)a)a)a + ((2p^2 g_1 - 2g_1)a_1\omega^2 b^4 + (((-8p^2 g_1 + 8g_1)b_1\omega^2 + (-8g_1 + 8p^2 g_1 + 4p_1 g_1 p\omega)\omega b)b^3 + ((4g_1 - 4p^2 g_1)a_1\omega^2 b^2 + (((-8p^2 g_1 + 8g_1)b_1\omega^2 + (-16g_1 + 16p^2 g_1 + 8p_1 g_1 p\omega)\omega b)b + ((-6p^2 g_1 + 6g_1)a_1\omega^2 + (-8g_1 + 8p^2 g_1 + 4p_1 g_1 p\omega)\omega a)a)a)a)a + (((-2 + 2p^2)b_1 a_1\omega^2 + ((2 - 2p^2)a_1 - 2p_1 a_1 p\omega)\omega b)b^3 + ((((-4p^2 + 4)b_1^2 + (-2 + 2p^2)a_1^2)\omega^2 + (((-8 + 8p^2)b_1 + 4p_1 b_1 p\omega)\omega + (-4p^2 - 4\omega pp_1 + 4)b)b)b^2 + (((-6p^2 + 6)b_1 a_1\omega^2 + (4p^2 - 4)a_1\omega b)b + ((2 - 2p^2)a_1^2\omega^2 + (((-8 + 8p^2)b_1 + 4p_1 b_1 p\omega)\omega + (-8\omega pp_1 + 8 - 8p^2)b)b + (((-6 + 6p^2)a_1 + 2p_1 a_1 p\omega)\omega + (-4p^2 - 4\omega pp_1 + 4)a)a)a)a)a)g)g)g)g,$$

$$c_0 = ((-1 + p^2)p^2 b_1^2\omega^2 + (((2 - 2p^2)p^2 b_1 - 2p_1 b_1 p^3\omega)\omega + ((-1 + p^2)p^2 + (2p^3 p_1 + p_1^2 p^2\omega)\omega)b)b)b^2 + (((-2 + 2p^2)p^2 b_1 a_1\omega^2 + ((2 - 2p^2)p^2 a_1 + (-2p_1 - 2p_1 p^2)pa_1\omega)\omega b)b + (((-p^2 + 1)b_1^2 + (1 + (-2 + p^2)p^2)a_1^2)\omega^2 + (((-2 + (4 - 2p^2)p^2)b_1 + (4p_1 - 2p_1 p^2)pb_1\omega)\omega + (1 + (2p^2 - 3)p^2 + ((-2p_1 + 4p_1 p^2)p + 2p_1^2 p^2\omega)\omega)b)b + (((-2 + (4 - 2p^2)p^2)a_1 + (2p_1 - 2p_1 p^2)pa_1\omega)\omega + (1 + (-2 + p^2)p^2 + ((-2p_1 + 2p_1 p^2)p + p_1^2 p^2\omega)\omega)a)a)a)a + (((2g_1 - 2p^2 g_1)p^2 b_1\omega^2 + ((2p^2 g_1 - 2g_1)p^2 + (2p_1 g_1 + 2p_1 g_1 p^2)p\omega)\omega b)b^3 + ((-2g_1 + (4g_1 - 2p^2 g_1)p^2)a_1\omega^2 b^2 + (((2g_1 - 2p^2 g_1)p^2 b_1\omega^2 + (2g_1 + (-6g_1 + 4p^2 g_1)p^2 + 4p_1 g_1 p^3\omega)\omega b)b + ((-2g_1 + (4g_1 - 2p^2 g_1)p^2)a_1\omega^2 + (2g_1 + (-4g_1 + 2p^2 g_1)p^2 + (-2p_1 g_1 + 2p_1 g_1 p^2)p\omega)\omega a)a)a)a + (((-1 + (3 - 2p^2)p^2)b_1^2\omega^2 + (((2 + (-6 + 4p^2)p^2)b_1 + (-2p_1 + 4p_1 p^2)pb_1\omega)\omega + (-1 + (3 - 2p^2)p^2 + ((2p_1 - 4p_1 p^2)p + (g_1^2 + (-2p_1^2 - 2g_1^2 + g_1^2 p^2)p^2)\omega)\omega)b)b)b^2 + (((-2 + (6 - 4p^2)p^2)b_1 a_1\omega^2 + ((2 + (-6 + 4p^2)p^2)a_1 + 4p_1 a_1 p^3\omega)\omega b)b + (((-1 + p^2)b_1^2 +$$

$(-2+(4-2p^2)p^2)a_1^2)\omega^2+(((4+(-8+4p^2)p^2)b_1+(-6p_1+4p_1p^2)pb_1\omega)\omega+(-3+(7-4p^2)p^2+((6p_1-8p_1p^2)p+(2g_1^2+$
$(-4g_1^2-4p_1^2+2g_1^2p^2)p^2)\omega)\omega)b)b+(((4+(-8+4p^2)p^2)a_1+(-4p_1+4p_1p^2)pa_1\omega)\omega+(-2+(4-2p^2)p^2+((4p_1-4p_1p^2)p+$
$(g_1^2+(-2p_1^2-2g_1^2+g_1^2p^2)p^2)\omega)\omega)a)a)a)a+(((2g_1+(-4g_1+2p^2g_1)p^2)b_1\omega^2+(-2g_1+(4g_1-2p^2g_1)p^2+(2p_1g_1-$
$2p_1g_1p^2)p\omega)\omega b)b^3+((2g_1+(-4g_1+2p^2g_1)p^2)a_1\omega^2b^2+(((2g_1+(-4g_1+2p^2g_1)p^2)b_1\omega^2+(-4g_1+(8g_1-4p^2g_1)p^2+$
$(4p_1g_1-4p_1g_1p^2)p\omega)\omega b)b+((2g_1+(-4g_1+2p^2g_1)p^2)a_1\omega^2+(-2g_1+(4g_1-2p^2g_1)p^2+(2p_1g_1-2p_1g_1p^2)p\omega)\omega a)a)a)a+$
$(((1+(-2+p^2)p^2)b_1^2\omega^2+(((-2+(4-2p^2)p^2)b_1+(2p_1-2p_1p^2)pb_1\omega)\omega+(1+(-2+p^2)p^2+((-2p_1+2p_1p^2)p+$
$p_1^2p^2\omega)\omega)b)b)b^2+(((2+(-4+2p^2)p^2)b_1a_1\omega^2+((-2+(4-2p^2)p^2)a_1+(2p_1-2p_1p^2)pa_1\omega)\omega b)b+((1+(-2+$
$p^2)p^2)a_1^2\omega^2+(((-2+(4-2p^2)p^2)b_1+(2p_1-2p_1p^2)pb_1\omega)\omega+(2+(-4+2p^2)p^2+((-4p_1+4p_1p^2)p+2p_1^2p^2\omega)\omega)b)b+$
$(((-2+(4-2p^2)p^2)a_1+(2p_1-2p_1p^2)pa_1\omega)\omega+(1+(-2+p^2)p^2+((-2p_1+2p_1p^2)p+p_1^2p^2\omega)\omega)a)a)a)a)g)g)g)g.$

Let ρ subsequently pass over all real roots, the searched unknowns k and d are defined by following expressions:

$k=\rho,$

$d=-\omega(((-1+p^2)p^2b_1\omega+((1-p^2)p^2+(\rho p^2a_1-p_1p^3)\omega+\rho^2p^2b^2)b)b^3+(((-1+p^2)p^2a_1\omega+(-2\rho\omega p^2b_1+2\rho p^2b)b)b^2+$
$(((-1+p^2)p^2b_1\omega+((2-2p^2)p^2-2p_1\omega p^3+3\rho^2p^2b^2)b)b+((-1+p^2)p^2a_1\omega+(-2\rho\omega p^2b_1+4\rho p^2b)b+((1-p^2)p^2+(-p_1p^3-$
$\rho p^2a_1)\omega+3\rho^2p^2b^2+(2\rho p^2+\rho^2p^2a)a)a)a)a)a+((g_1-g_1p^2)p^2\omega b^4+(2\rho\omega g_1b^4p^2+((-2g_1p^2+2g_1)p^2\omega b^2+(4\rho\omega g_1b^2p^2+$
$((g_1-g_1p^2)p^2\omega+2\rho\omega g_1p^2a)a)a)a)a+(((-1+(3-2p^2)p^2)b_1\omega+(1+(-3+2p^2)p^2+((p_1+2p_1p^2)p+(-\rho-2\rho p^2)a_1)\omega+$
$(-2\rho^2\omega b_1+(-3\rho^2p^2+\rho^2)b)b)b)b^3+(((-3+(5-2p^2)p^2)a_1\omega+(2\rho\omega p^2b_1+(-2\rho p^2+(2\rho pp_1-6\rho^2a_1)\omega-2\rho^3b^2)b)b)b^2+$
$(((3+(-1-2p^2)p^2)b_1\omega+((-4+4p^2)p^2+((2p_1+4p_1p^2)p+(-2\rho p^2-6\rho)a_1)\omega+(4\rho^2\omega b_1+(-9\rho^2p^2-3\rho^2)b)b)b)b+((1+$
$(-2p^2+1)p^2)a_1\omega+((2\rho p^2+8\rho)b_1\omega+(-4\rho p^2-4\rho+(4\rho pp_1-4\rho^2a_1)\omega-6\rho^3b^2)b)b+(-1+(-1+2p^2)p^2+((p_1+2p_1p^2)p+$
$3\rho a_1)\omega+(6\rho^2\omega b_1+(-9\rho^2p^2-9\rho^2)b)b+(-4\rho-2\rho p^2+(2\rho pp_1+2\rho^2a_1)\omega-6\rho^3b^2+(-3\rho^2p^2-5\rho^2-2\rho^3a)a)a)a)a)a)a+$
$(((3g_1+(-4g_1+g_1p^2)p^2)\omega+4\rho^2g_1\omega b^2)b^4+(4\rho g_1\omega b^4+(((2g_1+(-4g_1+2g_1p^2)p^2)\omega+4\rho^2g_1\omega b^2)b^2+((-g_1+g_1p^4)\omega-$
$4\rho^2g_1\omega b^2+(-4\rho g_1\omega-4\rho^2g_1\omega a)a)a^2)a)a+(((-1+p^2)p^2b_1\omega+((1-p^2)p^2+((-2p_1-p_1p^2)p+(\rho p^2+2\rho)a_1)\omega+(4\rho^2\omega b_1+$
$(3\rho^2p^2-\rho^2)b)b)b)b^3+(((2+(-3+p^2)p^2)a_1\omega+((2\rho p^2-4\rho)b_1\omega+(4\rho-2\rho p^2+(-4\rho pp_1+12\rho^2a_1)\omega+6\rho^3b^2)b)b)b^2+$
$(((-4+(3+p^2)p^2)b_1\omega+(2-2p^4+((-4p_1-2p_1p^2)p+(4\rho+4\rho p^2)a_1)\omega+(-8\rho^2\omega b_1+(9\rho^2p^2+9\rho^2)b)b)b)b+((-2+$
$(1+p^2)p^2)a_1\omega+((-12\rho+2\rho p^2)b_1\omega+(12\rho-4\rho p^2+(-8\rho pp_1+8\rho^2a_1)\omega+18\rho^3b^2)b)b+(2+(-1-p^2)p^2+((-2p_1-$
$p_1p^2)p+(3\rho p^2-6\rho)a_1)\omega+(-12\rho^2\omega b_1+(21\rho^2+9\rho^2p^2)b)b+(8\rho-2\rho p^2+(-4\rho^2a_1-4\rho pp_1)\omega+18\rho^3b^2+(11\rho^2+3\rho^2p^2+$
$6\rho^3a)a)a)a)a)a)a+(((g_1-g_1p^2)\omega-4\rho^2g_1\omega b^2)b^4+((-2\rho g_1p^2+4\rho g_1)\omega b^4+(((-2g_1p^2+2g_1)\omega-4\rho^2g_1\omega b^2)b^2+$
$((-4\rho g_1p^2+8\rho g_1)\omega b^2+((g_1-g_1p^2)\omega+4\rho^2g_1\omega b^2+((-2\rho g_1p^2+4\rho g_1)\omega+4\rho^2g_1\omega a)a)a)a)a)a+(((1-p^2)b_1\omega+(-1+$
$p^2+(-\rho a_1+pp_1)\omega+(-2\rho^2\omega b_1+(-\rho^2p^2-\rho^2)b)b)b)b^3+(((1-p^2)a_1\omega+((4\rho-2\rho p^2)b_1\omega+(2\rho p^2-4\rho+(2\rho pp_1-$
$6\rho^2a_1)\omega-6\rho^3b^2)b)b)b^2+(((1-p^2)b_1\omega+(-2+2p^2+(2pp_1+(2\rho-2\rho p^2)a_1)\omega+(4\rho^2\omega b_1+(-9\rho^2-3\rho^2p^2)b)b)b)b+((1-$
$p^2)a_1\omega+((4\rho-2\rho p^2)b_1\omega+(4\rho p^2-8\rho+(4\rho pp_1-4\rho^2a_1)\omega-18\rho^3b^2)b)b+(-1+p^2+(pp_1+(-2\rho p^2+3\rho)a_1)\omega+(6\rho^2\omega b_1+$
$(-15\rho^2-3\rho^2p^2)b)b+(2\rho p^2-4\rho+(2\rho pp_1+2\rho^2a_1)\omega-18\rho^3b^2+(-\rho^2p^2-7\rho^2-6\rho^3a)a)a)a)a)a)a+(\rho^2b^6+(2\rho^3b^6+$
$(3\rho^2b^4+(6\rho^3b^4+(3\rho^2b^2+(6\rho^3b^2+(\rho^2+2\rho^3a)a)a)a)a)a)a)g^2)g)g)g)g)g)g)/\rho/((-p^2\omega b_1+p^2b)b^4+(-2\omega b^3p^2a_1+$
$(2p^2b^3+(-2\omega bp^2a_1+(p^2b+p^2\omega b_1)a)a)a)a+(2\omega b^5g_1p^2+(4\omega b^3g_1p^2+2\omega bg_1p^2a^2)a^2+((-b_1\omega+(1+(-2\rho a_1+2pp_1)\omega-$
$2\rho^2b^2)b)b^4+(((-4+2p^2)a_1\omega+(6\rho b_1\omega-6\rho b)b)b^3+(((-2p^2+6)b_1\omega+(-2+(4\rho a_1+4pp_1)\omega-6\rho^2b^2)b)b^2+(((4+$
$2p^2)a_1\omega+(4\rho b_1\omega-12\rho b)b)b+((-1-2p^2)b_1\omega+(-3+(2pp_1+6\rho a_1)\omega-6\rho^2b^2)b+(-2\rho b_1\omega-6\rho b-2\rho^2ba)a)a)a)a)a+$
$(4g_1\omega b^5+(-8\rho\omega b^5g_1+(-16\rho\omega b^3g_1+(-4g_1\omega b-8\rho\omega bg_1a)a)a^2)a+(((3p^2-2)b_1\omega+(-3p^2+2+(-4pp_1+4\rho a_1)\omega+$
$6\rho^2b^2)b)b^4+((2a_1\omega p^2+(-12\rho b_1\omega+12\rho b)b)b^3+(((-8+4p^2)b_1\omega+(-6p^2+8+(-8\rho a_1-8pp_1)\omega+18\rho^2b^2)b)b^2+(((2p^2-$
$8)a_1\omega+(-8\rho b_1\omega+24\rho b)b)b+((2+p^2)b_1\omega+(6-3p^2+(-4pp_1-12\rho a_1)\omega+18\rho^2b^2)b+(4\rho b_1\omega+12\rho b+6\rho^2ba)a)a)a)a)a+$
$((4g_1-2g_1p^2)\omega b^5+(8\rho\omega b^5g_1+((8g_1-4g_1p^2)\omega b^3+(16\rho\omega b^3g_1+((4g_1-2g_1p^2)\omega b+8\rho\omega bg_1a)a)a)a)a+(((3-$
$2p^2)b_1\omega+(-3+2p^2+(-2\rho a_1+2pp_1)\omega-6\rho^2b^2)b)b^4+(((4-2p^2)a_1\omega+(6\rho b_1\omega-6\rho b)b)b^3+(((2-2p^2)b_1\omega+(-6+$
$4p^2+(4\rho a_1+4pp_1)\omega-18\rho^2b^2)b)b^2+(((4-2p^2)a_1\omega+(4\rho b_1\omega-12\rho b)b)b+(-b_1\omega+(-3+2p^2+(2pp_1+6\rho a_1)\omega-$
$18\rho^2b^2)b+(-2\rho b_1\omega-6\rho b-6\rho^2ba)a)a)a)a)a+(2\rho^2b^7+(6\rho^2b^5+(6\rho^2b^3+2\rho^2ba^2)a^2)a^2)g^2)g)g)g)g)g)g).$