# COMPUTING VALUE SETS FROM ONE POINT OF FREQUENCY RESPONSE

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Abstract: The aim of this paper is to present an explicit parametrization of the value set boundary for a model family consistent both with the process-control-oriented *a priori* information about the class of candidate models and with one experimentally obtained point of the process frequency response. More precisely, it is assumed that the real process can be described by a multiple fractional order pole model with or without a restriction on the total model order. The result obtained is further extended to the case where more points of a process frequency response are available. The presented results have important applications in design of robust controllers, in particular in the field of automatic tuning procedures.

Keywords: process control, uncertain linear systems, frequency response methods, fractal systems, interpolation, frequency domains

## 1. INTRODUCTION

It is well known from the classical control theory that process controllers can be designed on the base of a few points of the process frequency response. In this direction, the limit method is the popular Ziegler-Nichols frequency method which uses only one so called ultimate point. However, these traditional methods are neither systematic nor guarantee fulfillment of design specifications for an exactly given class of process transfer functions.

At present, when designing a robust controller in the frequency domain, it is usually assumed that the frequency response of the "true" system lies for each frequency on regions of the complex plane (Rotstein *et al.* 1998). These regions, called value sets, quantify the amount of model uncertainty at a given frequency. The problem of obtaining hard bounds of the value sets from experimental data has been considered before, see e.g. (Helmicki *et al.* 1991, Milanese *et al.* 1996). Although some of these methods are very sophisticated and general, none of them gives tight bounds for the case where only one or two frequency response points are available because of the minimum *a priori* information assumed.

In this paper, a novel approach to the above problem is presented. The key feature in the problem formulation is a new form of *a priori* information about the class of candidate models. In accordance with the majority of works in the process control field, it is assumed that the real process can be described by a multiple fractional order pole model (Charef *et al.* 1992, Podlubny 1999) in the form

$$P(s) = \frac{K}{\left(\prod_{i=1}^{p-1} (\tau_i s + 1)^{n_i}\right) s^{n_p}}, \qquad (1)$$

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where $p$ is an arbitrary integer and $K, \tau_i,\ i = 1, 2, \ldots, p-1,\ n_i,\ i = 1, 2, \ldots, p$ are non-negative real numbers. It is important to note that the class of all transfer functions (1) includes all integer order lag/dead time processes because of no restriction on the total order of (1). Moreover, it follows from (Skogestad 2003) that any lag/dead time transfer function with positive and/or negative numerator time constants may be also approximated (at least for the purpose of control design) by a transfer function in the form (1). Consequently, it is believed that the assumed class (1) is sufficiently rich for the process control purposes. Nevertheless, it should be explained, why the fractional order systems are introduced and preferred to common integer order transfer functions. The reason is clear from the previous paper (Schlegel 2002) on the same topics, where only the integer order systems are assumed. In that case, the characterization of the value set is much more complex because of discontinuity in the pole orders. Moreover, the fractional order systems seem to be a more appropriate representation of reality.

The main result of this paper is an explicit parametrization of the value set boundary for the model family consistent with the *a priori* information (1) and one experimentally obtained point of the process frequency response. This result is further extended to the case where more points of the process frequency response are available. However, then only the hard bounds of the value set instead of the exact boundary can be computed. The presented results have important applications in design of robust controllers and automatic tuning procedures. For example, an exact revision of all existing PID relay autotuners can be made as outlined in Section 5.

The paper is organized as follows. The frequency response interpolation problem is formulated in Section 2. The main result of this paper dealing with one point interpolation is given in Section 3. Section 4 deals with the case where more points of the process frequency response are available. Applications to robust controller design are outlined in Section 5. A sketch of the proof of the main theorem can be found in Section 6. Section 7 contains concluding remarks.

## 2. PROBLEM FORMULATION

In this paper, a class of fractional order systems in the form (1) is considered, where $K > 0$; $\tau_i > 0$, $n_i \geq m,\ i = 1, 2, \ldots, p-1,\ n_p$ is either 0 or $n_p \geq m$ and $\sum_{i=1}^{p} n_i \leq n$. It is assumed that $m$ and $n,\ 0 < m < n$ are given real numbers specifying the minimum admissible order of a fractional pole and the maximum admissible order of the whole process, respectively. In the following, this class will be denoted by $\mathcal{S}^{n,m}$. Note that the span of the class $\mathcal{S}^{n,m}$ may be controlled by a proper choice of the real numbers $m, n$. For example, if $m = 1$ and $n \to \infty$, then the set $\mathcal{S}^{n,m}$ contains all integer order lag systems with dead time. Consequently, it is believed, that the class $\mathcal{S}^{n,m}$ is suitable for description of the *a priori* process knowledge.

A subset of $\mathcal{S}^{n,m}$ containing all transfer functions $P \in \mathcal{S}^{n,m}$ which satisfy the interpolation conditions

$$P(\mathrm{j}\omega_i) = p_i, \qquad 0 > \arg P(\mathrm{j}\omega_i) > -2\pi,\ i = 1, 2, \ldots, l \tag{2}$$

where $p_i \in \mathbf{C}$ and $0 < \omega_1 < \omega_2 < \cdots < \omega_l$ are given, is denoted by $\mathcal{S}^{n,m}(\Pi), \Pi = \{(p_i, \omega_i)\}_{i=1}^{l}$, and called the *unfalsified plant family*. Similarly, the transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is called *unfalsified.* For $l = 1$ the symbol $\mathcal{S}^{n,m}(p_1, \omega_1)$ will be used instead of $\mathcal{S}^{n,m}(\Pi)$. Thus, if $P \in \mathcal{S}^{n,m}(p_1, \omega_1)$, then the transfer function $P$ must have the form (1) with the above constraints and its Nyquist plot $P(\mathrm{j}\omega)$ is required to pass through the given point $p_1$ at the frequency $\omega_1$ during the first circulation around the origin of the complex plane. Point out that the data $\Pi$ are assumed to be obtained from an identification experiment. In the following, the key concept is the value set, which is defined for any $\omega$ by

$$\mathcal{V}_\omega^{n,m}(\Pi) = \{P(\mathrm{j}\omega) :\ P \in \mathcal{S}^{n,m}(\Pi)\}. \tag{3}$$

The boundary of the value set $\mathcal{V}_\omega^{n,m}(\Pi)$ will be denoted by $\partial\mathcal{V}_\omega^{n,m}(\Pi)$. The main aim of this paper is to present an exact description of the boundary $\partial\mathcal{V}_\omega^{n,m}(\Pi)$ for the case $l = 1$ and give its not very conservative estimate for the case $l \geq 2$. For this purpose, the concept of an ultimate transfer function is introduced.

*Definition 1.* The unfalsified transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is called ultimate if there exists $\omega \notin \{\omega_1, \omega_2, \ldots, \omega_l\}$ such that $P(\mathrm{j}\omega) \in \partial\mathcal{V}_\omega^{n,m}(\Pi)$.

In other words, $P \in \mathcal{S}^{n,m}(\Pi)$ is ultimate if $P(\mathrm{j}\omega)$ creates a boundary point of $\mathcal{V}_\omega^{n,m}(\Pi)$ for at least one frequency $\omega > 0$ different from interpolating frequencies $\omega_1, \omega_2, \ldots, \omega_l$. It will be shown later that if $P$ generates a boundary point for one such frequency, then $P$ generates a boundary point for all $\omega > 0$. Thus, in our context, the concept of an ultimate transfer function is independent of the frequency. Consequently, the parametrization of all ultimate transfer functions is sufficient for exact description of the boundary $\partial\mathcal{V}_\omega^{n,m}(\Pi)$.

## 3. ONE POINT INTERPOLATION

This section deals with the simplest case when the experimental data contain only one sample of the frequency response. The next theorem gives an exact parametrization of all ultimate transfer functions for this case.

*Theorem 1.* Let $\Pi = \{(p_1, \omega_1)\}$, $p_1 = r\mathrm{e}^{-\mathrm{j}\varphi}$, $r > 0$, $0 < \varphi < 2\pi$, $\omega_1 > 0$ and $\varphi \le (n-m)\frac{\pi}{2}$. Then $\mathcal{S}^{n,m}(p_1, \omega_1) \ne \emptyset$ and the unfalsified transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is ultimate if and only if it can be expressed in one of the following forms

$$(i) \qquad P_1(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n_1(\alpha)}}, \tag{4}$$

where

$$\tau_1(\alpha) = \frac{1}{\omega_1}\tan\alpha,$$
$$n_1(\alpha) = \frac{\varphi}{\alpha},$$
$$K(\alpha) = r\sqrt{(\tau_1^2(\alpha)\omega_1^2+1)^{n_1(\alpha)}}$$

and $\alpha$ is sweeping the interval

$$I_1 = \left[\frac{\varphi}{n}, \min\left\{\frac{\pi}{2}, \frac{\varphi}{m}\right\}\right]$$

$$(ii)\ P_2(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n-m}(\tau_2(\alpha)s+1)^m}, \tag{5}$$

where

$$\tau_1(\alpha) = \frac{1}{\omega_1}\tan\alpha,$$
$$\tau_2(\alpha) = \frac{1}{\omega_1}\tan\frac{\varphi-(n-m)\alpha}{m},$$
$$K(\alpha) = r\sqrt{(\tau_1^2(\alpha)\omega_1^2+1)^{n-m}(\tau_2^2(\alpha)\omega_1^2+1)^m}$$

and $\alpha$ is sweeping the interval

$$I_2 = \left[\max\left\{0, \frac{\varphi - m\frac{\pi}{2}}{n-m}\right\}, \frac{\varphi}{n}\right]$$

$$(iii)\ P_3(s,\alpha) = \frac{K(\alpha)}{(\tau_1(\alpha)s+1)^{n-n_2(\alpha)}s^{n_2(\alpha)}}, \tag{6}$$

where

$$\tau_1(\alpha) = \frac{1}{\omega_1}\tan\alpha,$$
$$n_2(\alpha) = \frac{\varphi - n\alpha}{\frac{\pi}{2}-\alpha},$$
$$K(\alpha) = r\sqrt{(\tau_1^2(\alpha)\omega_1^2+1)^{n-n_2(\alpha)}}$$

and $\alpha$ is sweeping the interval

$$I_3 = \left[0, \frac{\varphi - m\frac{\pi}{2}}{n-m}\right].$$

Moreover, the value set $\mathcal{V}_\omega^{n,m}(p_1,\omega_1)$, $\omega > 0$, $\omega \ne \omega_1$ is a closed domain bounded by three arcs: $P_1(\mathrm{j}\omega,\alpha), \alpha \in I_1$; $P_2(\mathrm{j}\omega,\alpha), \alpha \in I_2$ and $P_3(\mathrm{j}\omega,\alpha)$, $\alpha \in I_3$. □

Several remarks can be made regarding Theorem 1.

*Remark 1.* The version of the Theorem 1 for the case where no *a priori* restriction on the process order is considered, can be simply obtained by the limiting process $n \to \infty$.

*Remark 2.* For the detail analysis of the value set $\mathcal{V}_\omega^{n,m}(r\mathrm{e}^{-\mathrm{j}\varphi}, \omega_1)$, we can restrict ourselves without loss of generality to the case $r = 1$ and $\omega_1 = 1$ because of the normalization in gain and time. Consequently, all illustrating figures given below will be for this case (Fig. 1, 2 and 3).

*Remark 3.* Hereafter, the endpoints of the arcs $P_i(\mathrm{j}\omega,\alpha)$, $\alpha \in I$, $i = 1,2,3$, are called vertices of the value set $\mathcal{V}_\omega^{n,m}(p_1,\omega_1)$ (see Fig. 1). It follows from Theorem 1 that they are generated by the transfer functions

$$V_1(s) = \lim_{\alpha\to\alpha_1^-} P_1(s,\alpha) = \lim_{\alpha\to\alpha_2^+} P_2(s,\alpha),$$
$$V_2(s) = \lim_{\alpha\to\alpha_1^+} P_1(s,\alpha) = \lim_{\alpha\to\alpha_3^-} P_3(s,\alpha),$$
$$V_3(s) = \lim_{\alpha\to\alpha_2^-} P_2(s,\alpha) = \lim_{\alpha\to\alpha_3^+} P_3(s,\alpha),$$

where $\alpha_i^-$ is the begin and $\alpha_i^+$ end point of the interval $I_i$, $i = 1,2,3$. For the case $0 < \varphi < m\frac{\pi}{2}$, $r = 1$, $\omega_1 = 1$ we obtain (Fig. **??**)

$$V_1(s) = \frac{\sqrt{\left(\tan^2\frac{\varphi}{n}+1\right)^n}}{\left(\tan\frac{\varphi}{n}s+1\right)^n}$$
$$V_2(s) = V_3(s) = \frac{\sqrt{\left(\tan^2\frac{\varphi}{m}+1\right)^m}}{\left(\tan\frac{\varphi}{m}s+1\right)^m}. \tag{7}$$

For the case $\varphi \ge m\frac{\pi}{2}$, $r = 1, \omega_1 = 1$, $V_1(s)$ is also given by (7) and

$$V_2(s) = \frac{1}{s^{\frac{2\varphi}{\pi}}}$$
$$V_3(s) = \frac{\sqrt{\left(\tan^2\frac{\varphi-m\frac{\pi}{2}}{n-m}+1\right)^{n-m}}}{\left(\tan\frac{\varphi-m\frac{\pi}{2}}{n-m}s+1\right)^{n-m}s^m} \tag{8}$$

Example of this case is shown if Fig. 2. Particularly interesting forms of the vertex transfer functions are obtained in the limit case $n \to \infty$, $\varphi \ge m\frac{\pi}{2}, r = 1, \omega_1 = 1$. Here

$$V_1(s) = \mathrm{e}^{-\varphi s}, \tag{9}$$
$$V_2(s) = \frac{1}{s^{\frac{2\varphi}{\pi}}}, \tag{10}$$
$$V_3(s) = \frac{\mathrm{e}^{-(\varphi - m\frac{\pi}{2})s}}{s^m}. \tag{11}$$

Thus, the vertices are generated by a dead time, fractional order integrator and a fractional order integrator plus dead time. A challenging question

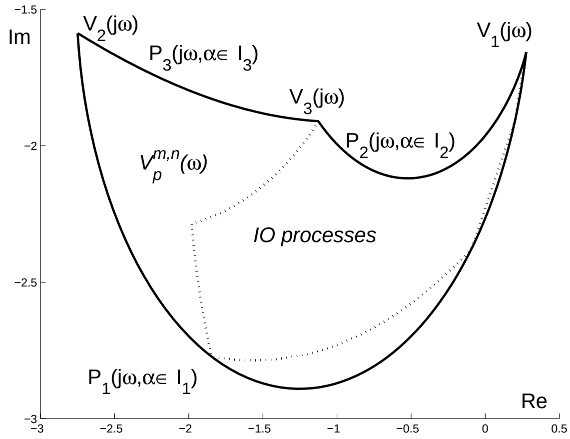


Fig. 1. The value set $\mathcal{V}_{0.5}^{5,1}(\mathrm{e}^{-\mathrm{j}5/6\pi}, 1)$ - solid line; the corresponding value set of all unfalsified integer order models - dotted line.

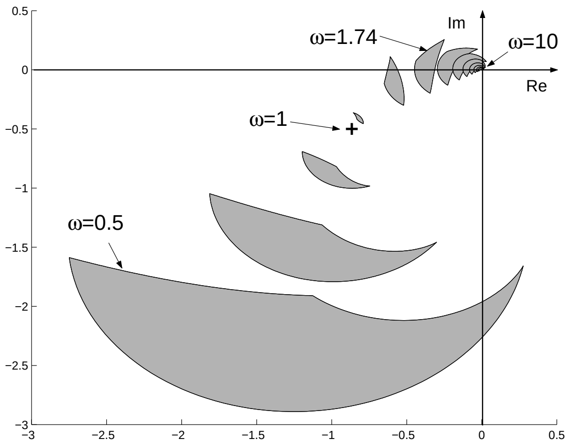


Fig. 2. The value sets $\mathcal{V}_{\omega}^{5,1}(\mathrm{e}^{-\mathrm{j}5/6\pi}, 1)$ for several $\omega$.

arises. Does there exist a robust controller with a satisfactory performance for all three vertex systems (9-11), where $m$ is e. g. 1 and $\varphi$ is appropriately chosen? In Section 5, we give an answer for the special case of the PI controller.

*Remark 4.* The span of the admissible plant family $\mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi}, 1),\ 0 < \varphi < 2\pi$ can be controlled by a proper choice of the real numbers $m$ and $n$. For example, the value set $\mathcal{V}_{\omega}^{n,m}(p_1, \omega_1)$ enlarges if $n$ is increasing and $m$ is decreasing (Fig. 3).

## 4. MULTIPLE POINT INTERPOLATION

In this section, it is shown that Theorem 1 can be simply used for obtaining hard bounds of the value set $\mathcal{V}_{\omega}^{n,m}(\Pi)$ when $\Pi$ contains two or more frequency response samples.

Let $\Pi = \{(p_1, \omega_1), (p_2, \omega_2)\}\ ,\ \omega_1 < \omega_2$, then from the definition of the value set (3) it follows that $\mathcal{S}^{n,m}(\Pi) \neq \emptyset$ if and only if $p_1 \in \mathcal{V}_{\omega_1}^{n,m}(p_2, \omega_2)$. Thus, using Theorem 1, it can be simply checked whether the experimental data $\Pi$ are consistent

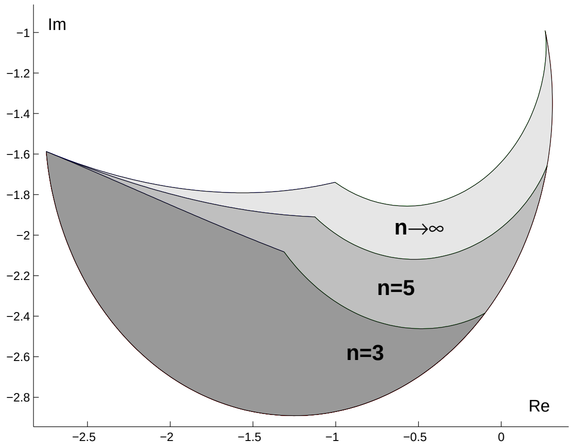


Fig. 3. Shaping of the value set $\mathcal{V}_{0.5}^{n,1}(\mathrm{e}^{-\mathrm{j}5/6\pi}, 1)$ by upper bound $n$ of the model order.

with the *a priori* assumption on the process transfer function. Moreover, any point of $\mathcal{V}_{\omega}^{n,m}(\Pi)$ must fall both into $\mathcal{V}_{\omega}^{n,m}(p_1, \omega_1)$ and $\mathcal{V}_{\omega}^{n,m}(p_2, \omega_2)$ for any $\omega$ as follows again directly from the definitions. Consequently, it holds that

$$\mathcal{V}_{\omega}^{n,m}(\Pi) \subset \mathcal{V}_{\omega}^{n,m}(p_1, \omega_1) \cap \mathcal{V}_{\omega}^{n,m}(p_2, \omega_2). \qquad (12)$$

Since Theorem 1 gives an effective method for computing the boundaries $\partial\mathcal{V}_{\omega}^{n,m}(p_i, \omega_i),\ i = 1, 2$, the boundary of the intersection involved in (12) can be simply computed and the hard bounds of the value set $\mathcal{V}_{\omega}^{n,m}(\Pi)$ determined. Note that (12) can be extended to an arbitrary number of interpolation conditions (2).

*Example 1.* Consider the case where two frequency response samples $\Pi = \{(0.5\,\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}, 0.3), (0.07\mathrm{e}^{-\mathrm{j}\pi}, 1.18)\}$ are available. The intersection of the value sets (12) is depicted in Fig. 4. for two frequencies from the interval $(\omega_1, \omega_2)$. Notice that this intersection is much smaller than the value sets for each individual point.

## 5. ROBUST CONTROLLER DESIGN

In this section, Theorem 1 and the robustness regions method (Shafiei and Shenton 1997, Astrom and Hagglund 2001) will be employed in design of the robust PI controller when only one point $(p_1, \omega_1)$ of the process frequency response is available. Suppose that $p_1 = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$ and $\omega_1 = 1$. Then, according to Theorem 1 and Remark 3, the value set $\mathcal{V}_{\infty}^{\infty,1}(\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}, 1)$ has only the two vertices generated by the transfer functions (9−11)

$$V_1(s) = \mathrm{e}^{-\frac{\pi}{2}s}, \qquad (13)$$

$$V_2(s) = V_3(s) = \frac{1}{s}. \qquad (14)$$

Further, the PI controller is assumed in the form

$$C(s) = k + \frac{k_i}{s}. \qquad (15)$$

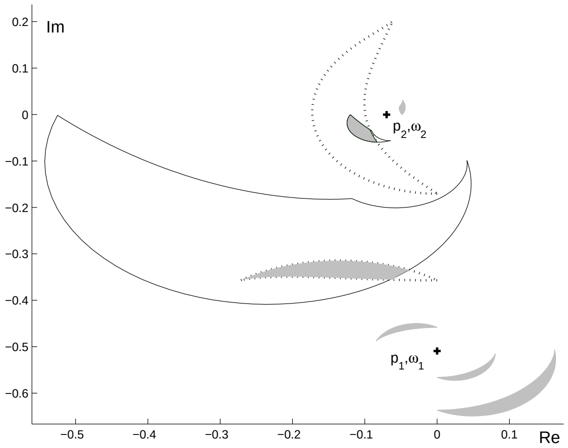


Fig. 4. Computing hard bounds of the value set $\mathcal{V}_{\omega}^{n,m}(\Pi)$ for the case $l = 2$, $p_1 = 0.5\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$, $\omega_1 = 0.30$, $p_2 = 0.07\mathrm{e}^{-\mathrm{j}\pi}$, $\omega_2 = 1.18$, $\omega = \{0.24, 0.27, 0.34, 0.40, 0.90, 1.41\}$; $\mathcal{V}_{\omega}^{n,m}(p_1, \omega_1)$ - dotted line; $\mathcal{V}_{\omega}^{n,m}(p_2, \omega_2)$ - solid line.

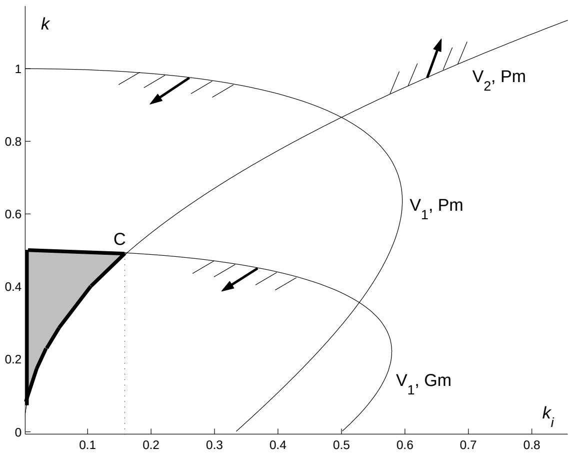


Fig. 5. The robustness regions for vertex processes (13-14).

The first step is to design the parameters $k, k_i$ such that the gain and phase margins are at least 2 and $60^o$, respectively, for both process transfer functions (13−14). Corresponding robustness regions in the parameter plane $k - k_i$ are depicted in Fig. 5. Any point from their intersection (the shaded region in Fig. 5) fulfills the above design specifications. Moreover, the point with the maximum $k_i$ coordinate minimizes the criterion $\int_0^{\infty} e(t)\mathrm{d}t$ (Astrom and Hagglund 2001). Choosing this point, $k = 0.48$, $k_i = 0.17$. To complete the robust design procedure, it must be checked whether all unfalsified transfer functions given by Theorem 1 also fulfill the design specifications. However, in our case this fact is apparent from Fig. 6.

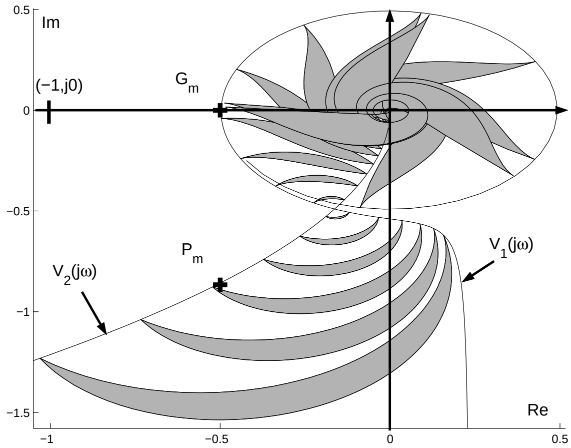


Fig. 6. The value sets compensated by the designed PI controller.

## 6. PROOF OF THEOREM 1

The most important tool in the proof of Theorem 1 is the Implicit Function Theorem, known from calculus (Nijmeijer and Schaft 1990). Without loss of generality the proof can be restricted to the case $r = 1$, $\omega_1 = 1$.

*Lemma 1.* If an unfalsified transfer function $P \in \mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi}, 1)$ has at least three mutually different negative poles, then $P$ is not ultimate.

*Proof.* Firstly, let $P \in \mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi}, 1)$ have just three mutually different negative poles, i.e.,

$$P(s) = \frac{K}{(\tau_1 s + 1)^{n_1}(\tau_2 s + 1)^{n_2}(\tau_3 s + 1)^{n_3}}, \quad (16)$$

where $\tau_i = \vartheta_i$, $i = 1, 2, 3$, $0 < \vartheta_1 < \vartheta_2 < \vartheta_3$, $K > 0$, $n_i \geq m$, $i = 1, 2, 3$, and $\sum_{i=1}^{3} n_i \leq n$. Define

$$h(\tau_1, \tau_2, \tau_3) = \sum_{i=1}^{3} n_i \arctan \tau_i. \quad (17)$$

By hypothesis

$$h(\vartheta_1, \vartheta_2, \vartheta_3) = \varphi \quad (18)$$

and

$$\frac{\partial}{\partial \tau_3} h(\vartheta_1, \vartheta_2, \vartheta_3) = \frac{n_3}{1 + \vartheta_3^2} \neq 0.$$

Thus, by the Implicit Function Theorem there exists a unique smooth solution

$$\tau_3 = \tau_3(\tau_1, \tau_2) \quad (19)$$

of $h(\tau_1, \tau_2, \tau_3) = \varphi$ defined in some neighborhood of the point $(\tau_1, \tau_2) = (\vartheta_1, \vartheta_2)$. Note that $\tau_3(\vartheta_1, \vartheta_2) = \vartheta_3$. Substituting (19) into (17) and by differentiating the result we obtain

$$\frac{\partial}{\partial \tau_1} \tau_3(\tau_1, \tau_2) = -\frac{n_1 \left(1 + \tau_3(\tau_1, \tau_2)^2\right)}{n_3 (1 + \tau_1)^2} \quad (20)$$

and

$$\frac{\partial}{\partial \tau_2} \tau_3(\tau_1, \tau_2) = -\frac{n_2 \left(1 + \tau_3(\tau_1, \tau_2)^2\right)}{n_3 (1 + \tau_2)^2}. \quad (21)$$

Now define

$$f_1(\tau_1,\tau_2) = -\arg P(\mathrm{j}\omega),$$
$$f_2(\tau_1,\tau_2) = |P(\mathrm{j}\omega)|^2,$$

where $P$ is given by (16). Note that the functions $f_1, f_2$ unambiguously determine the point $P(\mathrm{j}\omega)$ in the complex plane because they are its polar coordinates. From (16), (19) and $P \in \mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi},1)$, it follows that

$$f_1(\tau_1,\tau_2) = n_1 \arctan(\tau_1\omega) + n_2 \arctan(\tau_2\omega) + \\ +n_3 \arctan(\tau_3(\tau_1,\tau_2)\omega) \quad (22)$$

and

$$f_2(\tau_1,\tau_2) = \left(\frac{\tau_1^2+1}{\tau_1^2\omega^2+1}\right)^{n_1}\left(\frac{\tau_2^2+1}{\tau_2^2\omega^2+1}\right)^{n_2} \cdot \\ \cdot\left(\frac{\tau_3(\tau_1,\tau_2)^2+1}{\tau_3(\tau_1,\tau_2)^2\omega^2+1}\right)^{n_3}. \quad (23)$$

After tedious derivation, using (22), (23), (20) and (21) we obtain

$$J(\tau_1,\tau_2) = \det\begin{vmatrix} \dfrac{\partial f_1(\tau_1,\tau_2)}{\partial\tau_1} & \dfrac{\partial f_1(\tau_1,\tau_2)}{\partial\tau_2} \\ \dfrac{\partial f_2(\tau_1,\tau_2)}{\partial\tau_1} & \dfrac{\partial f_2(\tau_1,\tau_2)}{\partial\tau_2}\end{vmatrix} = \\ = \frac{2\omega(\omega^2-1)^2(\tau_1-\tau_2)(\tau_1-\tau_3)(\tau_2-\tau_3)}{(\tau_1^2+1)(\tau_1^2\omega^2+1)(\tau_2^2+1)(\tau_2^2\omega_2+1)} \cdot \\ \cdot\frac{n_1n_2|P(\mathrm{j}\omega)|^2}{\tau_3^2\omega^2+1}, \quad (24)$$

where $\tau_3 = \tau_3(\tau_1,\tau_2)$. Consequently, the Jacobian $J(\tau_1,\tau_2)$ is nonzero at the point $(\tau_1,\tau_2) = (\vartheta_1,\vartheta_2)$ for all frequencies $\omega > 0, \omega \neq 1$. Thus, by the Implicit Function Theorem, the point $P(\mathrm{j}\omega)$ for $(\tau_1,\tau_2) = (\vartheta_1,\vartheta_2)$ is an interior point of the value set $\mathcal{V}_\omega^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi},1)$ for all $\omega > 0, \omega \neq 1$ and the transfer function (16) is not ultimate.

Now, assume that $P$ is in the form

$$P(s) = \frac{K}{\prod_{i=1}^{3}(\tau_i s+1)^{n_i}} \cdot \frac{1}{\left(\prod_{i=4}^{p-1}(\tau_i s+1)^{n_i}\right)s^{n_p}} = \\ = \bar{P}(s)R(s), \quad (25)$$

where $\tau_i = \vartheta_i, \quad i = 1,2,3,\ 0 < \vartheta_1 < \vartheta_2 < \vartheta_3$. From the assumption $P(\mathrm{j}) = \mathrm{e}^{-\mathrm{j}\varphi}$ it follows that $\bar{P}(\mathrm{j}) = \frac{1}{R(\mathrm{j})}\mathrm{e}^{-\mathrm{j}\varphi} = \bar{p_1}$. We know from the first part of the proof that the point $\bar{P}(\mathrm{j}\omega)$ is an interior point of $\mathcal{V}_\omega^{n,m}(\bar{p_1},1)$. Thus also the point $P(\mathrm{j}\omega)$ is an interior point of $\mathcal{V}_\omega^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi},1)$ because of (25). Consequently, transfer function $P$ in the form (25) can not be ultimate. □

*Lemma 2.* If an unfalsified transfer function $P \in \mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi},1)$ is in the form

$$P(s) = \frac{K}{(\tau_1 s+1)^{n_1}(\tau_2 s+1)^{n_2}}, \quad (26)$$

where $K > 0,\ 0 < \tau_1 < \tau_2,\ n_1 > m$ and $n_1 + n_2 < n$, then $P$ is not ultimate.

*Lemma 3.* If an unfalsified transfer function $P \in \mathcal{S}^{n,m}(\mathrm{e}^{-\mathrm{j}\varphi},1)$ is in the form

$$P(s) = \frac{K}{(\tau_1 s+1)^{n-\nu}(\tau_2 s+1)^{\nu}}, \quad (27)$$

where $K > 0,\ \tau_1 > 0, \tau_2 > 0,\ \tau_1 \neq \tau_2,\ \nu > m$, then P is not ultimate.

Proofs of Lemma 2 and Lemma 3 are similar to the proof of Lemma 1 and are omitted for brevity. Using Lemmas 1-3, it is not difficult to complete the proof of Theorem 1.

## 7. CONCLUSIONS

In this paper, an explicit parametrization of the value set boundary for a multiple fractional order pole model has been presented under the assumption that only one point of the process frequency response is available. In particular, it is shown that the value sets are closed domains bounded by at most three smooth arcs. Furthermore, it has been demonstrated that this result can be simply used for obtaining hard bounds of the value sets if two or more points of the process frequency response are given. The presented results have important applications in design of robust controllers, in particular for automatic tuning procedures based on one or two points of a process frequency response.

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