GENERALIZED ROBUSTNESS REGIONS FOR PID CONTROLLERS

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Abstract: The problem of design a fixed PID controller for several LTI models is recognized as one of the fundamental problem in control theory. Recently, interesting results concerning the parameterization of all PID controllers stabilizing a given system were published. In this paper, the systematic procedure for construction of robustness regions in the parameter plane of the PID controller is presented when we require that the open loop Nyquist plot does not have an intersection with a given general conic. Further, it is shown how use this procedure for design of the fixed PID controller for several process models simultaneously.

Keywords: PID control, robustness design, simultaneous stabilization, parametric plane method.

1 INTRODUCTION

While modern control theory allows for a high level of sophistication in control system design, proportional-integral-derivative (PID) controllers are still widely used in the process industry.

Several methods of tuning PID controllers for satisfactory behaviour are used in practice. Most of them are only semi-empirical methods selected for specific problem requirements (Äström and Hägglund 1995). There exist very few really systematic procedures applicable for more complex systems such as non-minimum-phase systems, unstable systems and systems with significant time delay. One exception is the method based on the classical D-partition (Neimark 1948). Different modifications of this method for the design of the PID controller are described in (Shafiei and Shenton 1997), (Ho, Datta and Bhattacharyya 1996, 1997), (Munro, Löylemer and Baki 1999), (Äström and Hägglund 2001). The key concept in these methods is the stability or robustness region in the parameter plane. It turns out that much insight into PID control can be obtained by analyzing such regions.

The design technique presented in this paper gives a further step in this direction. The main result obtained isolates the robustness region in the PID parameter plane when we require that the open loop transfer function does not have an intersection with a given general conic. Moreover, the solution is completely analytical. For plotting the regions, only the procedure for determining roots of polynomials is needed.

The paper is organized as follows: In Section 2, the parameterization of the PID controller is introduced. In Section 3, the conic constrain on the open loop transfer function is formulated and interpreted in terms of gain and phase margins and of sensitivity and complementary sensitivity functions. The results obtained are used in several examples in Section 4. Finally, Section 5 constrains some concluding remarks.

2 PID CONTROL LOOP

Consider the PID feedback control system shown in Fig. 1, in which $P(s)$ represents the process with the transfer function

$$P(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+...+b_1 s + b_0}{s^n + a_{n-1}s^{n-1}+...+a_1 s + a_0}e^{-Ds} \qquad (1)$$

where $n \ge m$ and a_i, b_i, D are real coefficients and $C(s)$ represents the PID controller with the transfer function

$$C(s)=k\left(1+\frac{1}{T_i s}+T_d s\right)=k+\frac{k_i}{s}+k_d s \quad , \qquad (2)$$

where k, T_i and T_d are its engineering parameters and

$k_i = k/T_i$, $k_d = kT_d$ are gains used in the following.

To be able to use the D-partition method, we restrict the problem to the case of two design parameters k and d for which it holds

$$\begin{aligned} k_i &= kd \ , \\ k_d &= k\frac{f}{d}=\frac{fk^2}{k_i} \ , \end{aligned} \qquad (3)$$

where $d = 1/T_i$ and f is a fixed parameter with the meaning $f = T_d / T_i$. Classically it was quite common to postulate $f = 1/4$. Note, that the given point in $k - k_i$ plane determines unambiguously the gain k_d according to (3).

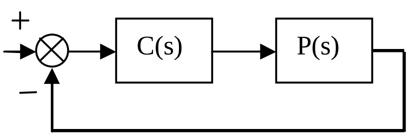


Figure 1. The PID feedback control system.

3 GENERALIZED ROBUSTNESS REGIONS

We want to find all pairs (k, k_i) in the parameter plane for which the Nyquist plot

$$L(jw) = C(jw)P(jw) = u + jv \qquad (4)$$

does not have an intersection with the conic V, but is tangential to it for some $w \in [0, +\infty)$. The conic V is arbitrary circle, parabola, ellipse or hyperbola defined by the general equation

$$Kq^2 + Qr^2 + Mqr + Nq + Or + R = 0 \ , \qquad (5)$$

where K, Q, M, N, O, R are real coefficients and fulfil the rules for the specific conics.

Denote for simplicity

$$C(jw) \equiv x + jy \ , \qquad (6)$$

$$P(jw) \equiv a + jb \ , \qquad (7)$$

where

$$x = k \ , \qquad (8)$$

$$y = k\left(\frac{fw}{d}-\frac{d}{w}\right) \qquad (9)$$

as it follows from (2) and (3). Now, Eq. (4) gives

$$\begin{aligned} u &= ak - bk\left(\frac{fw}{d}-\frac{d}{w}\right) \ , \\ v &= ak\left(\frac{fw}{d}-\frac{d}{w}\right)+bk. \end{aligned} \qquad (10)$$

Further denote

$$\begin{aligned} \frac{d}{dw}L(jw) &\equiv u_1 + jv_1 \ , \\ \frac{d}{dw}P(jw) &\equiv a_1 + jb_1 \ , \end{aligned} \qquad (11)$$

then the Nyquist plot $L(jw)$ is tangential to the conic V at the frequency w iff it holds

$$Ku^2 + Qv^2 + Muv + Nu + Ov + R = 0 \qquad (12)$$

and

$$(2Ku + Mv + N)u_1 + (2Qv + Mu + O)v_1 = 0. \qquad (13)$$

After substitution u, v, u_1, v_1 from (10) and (11), Eq. (13) gives

$$k(Ak + B) = 0 , \qquad (14)$$

where A, B are the known function of w and d. Obviously, the significant solution of (14) for fixed d is given by

$$k(d) = -\frac{B}{A}.$$

Using this, from Eq. (12) we obtain

$$\sum_{i=0}^{8} c_i d^i = 0 , \qquad (15)$$

where $c_i, i = 0,1,...,8$ are the known function of w. Let d_l, $l = 1,2,...,m$ denote the real roots of (15), then the pairs $(k(d_l), k_i \equiv k(d_l)d_l)$, $l = 1,2,...,m$ describe parametric curves with the parameter w

which create the boundaries of robustness regions in the parameter plane $k-k_i$. So the required boundary of the robustness region in the parameter plane $k-k_i$ may be computed exactly. In next two sections we describe the special results for the case when the conic degenerates to a point or when defines a circle.

3.1 *Gain and phase margin regions*

In the case when the conic in the complex plane degenerates to the point, the corresponding D-partition problem is quite simple. It is shown in (Shafiei and Shenton 1997), that solution of such problem for specific points leads to gain and phase margin regions. Here, the general case is treated for the purpose of arbitrary shaping of the open loop transfer function $L(jw)$. We want to find all points in the $k-k_i$ parameter plane for which the corresponding Nyquist plot crosses the point c, i.e. it holds

$$L(jw) = c \equiv u + jv \tag{16}$$

for some ω. After some calculations we obtain

$$\begin{aligned} k &= \frac{au+bv}{a^2+b^2} \ , \\ d &= \frac{w(-av+bu}{2(au+bv)} \pm \\ &\pm \frac{\sqrt{a^2v^2-2abuv+u^2b^2+4b^2v^2f+8abuvf+4a^2u^2f}}{2(au+bv)} \ . \end{aligned} \tag{17}$$

The parametric curves (17) divide the complex plain into several regions. All points of the given region fulfil the property that the point c lies on the left (or on the right) of the corresponding Nyquist plot $L(jw)$. If we plot such regions for several points c, we can isolate the region with $L(jw)$ properly shaped.

3.2 *Sensitivity and complementary sensitivity regions*

For the control loop shown in Fig. 1, the open loop transfer function $L(s)$, the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ are, respectively, defined by

$$\begin{aligned} L(s) &= C(s)P(s) \ , \\ S(s) &= \frac{1}{1+L(s)} \ , \\ T(s) &= \frac{L(s)}{1+L(s)} \ . \end{aligned} \tag{18}$$

The maximum sensitivity M_S and the maximum complementary sensitivity M_P are, respectively, defined by

$$M_S = \sup_w |S(jw)| \ , \quad M_P = \sup_w |T(jw)| \ .$$

The controller design, according to (Äström 1998), involves the determination of three controller gains (k, k_i, k_d) in such a way that the disturbance – rejection performance index

$$J = \frac{1}{k_i}$$

is minimized while the closed loop is stable and satisfy the specified sensitivity M_S and/or complementary sensitivity M_P. Both of these constraints can be interpreted as a limitation on the open loop $L(jw)$ in the following form: Nyquist plot $L(jw)$, $w \in [0,+\infty)$ is outside a circle with center $s=c$ and radius $r>0$. More formally, $L(jw) \notin U(c,r)$ for arbitary $w \ge 0$, where $U(c,r) \equiv \{s \in C : |s-c| < r\}$.

The sensitivity M_S corresponds with

$$c = -1, \quad r = \frac{1}{M_S} \ ,$$

and the complementary sensitivity M_P with

$$c = \frac{M_P^2}{1-M_P^2}, \quad r = \frac{M_P}{|M_P^2 - 1|}.$$

The Nyquist plot $L(jw)$ is tangential to the circle with the center c and radius r at the frequency w iff it holds

$$(u-c)^2 + v^2 = r^2 \tag{19}$$

and

$$(u-c)u_1 + vv_1 = 0 . \tag{20}$$

Using (10) and (11), we obtain from (19) and (20)

$$\left[ak - bk\left(\frac{fw}{d} - \frac{d}{w}\right) - c\right]^2 + \left[ak\left(\frac{fw}{d} - \frac{d}{w}\right) + bk\right]^2 = r^2 \tag{21}$$

and

$$\begin{aligned} &k\left[ak - bk\left(\frac{fw}{d} - \frac{d}{w}\right) - c\right]\left[a_1 - b_1\left(\frac{fw}{d} - \frac{d}{w}\right) - b\left(\frac{f}{d} + \frac{d}{w^2}\right)\right] + \\ &+ k^2\left[a\left(\frac{fw}{d} - \frac{d}{w}\right) + b\right]\left[a_1\left(\frac{fw}{d} - \frac{d}{w}\right) + a\left(\frac{f}{d} + \frac{d}{w^2}\right) + b_1\right] = 0 . \end{aligned} \tag{22}$$

Eq. (22) has explicit solution for indeterminate variable $k(d)$. By substituting this solution into (21), we obtain a polynomial equation for indeterminate variable d.

Let $d_l,\ l = 1, 2, ..., m$ denote the real roots of this equation, then the pairs $(k(d_l),\ k_i \equiv k(d_l)d_l),\ l = 1, 2, ..., m$ describe parametric curves with the parameter w which create the boundaries of robustness regions in the parameter plane $k - k_i$. If the point (k_p, k_i) moves from one region to another, then when it crosses the boundary there exists at least one frequency w for which hold (19) and (20) and therefore the plot $L(jw)$ is tangential to the circle considered and vice versa.

4 EXAMPLES

To illustrate the above PID design method we solve several examples.

Example 1. Consider the following process

$$F(s) = \frac{(0.5s+1)\,e^{-1.5s}}{(0.25s+1)^4}$$

and two conditions: maximum sensitivity requirement $M_S = 2.0$ and the parabola P with the top $0.0 + 0.3\mathrm{j}$ and focal distance $p = -1.0$ which is described by

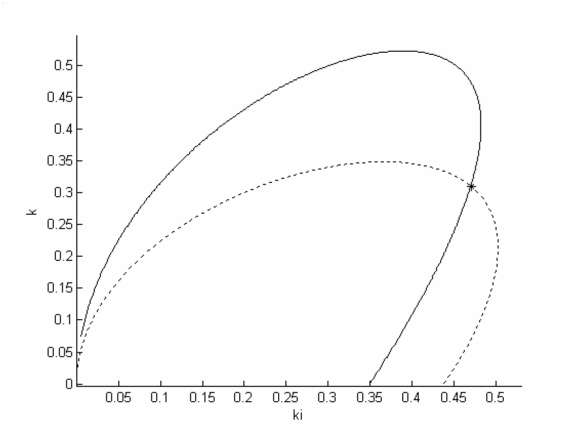
$$r = Kq^2 + Nq + R\,.$$

Now, from (12) and (13) we obtain

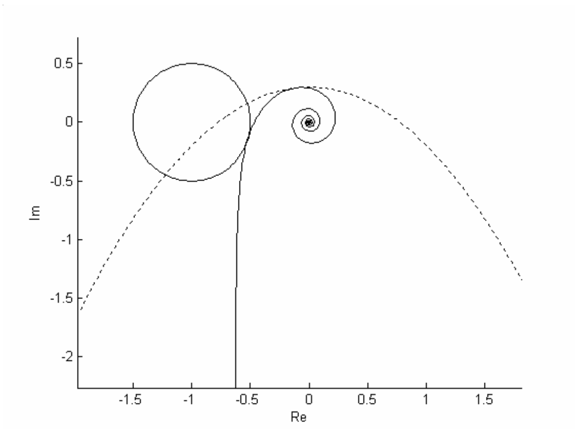
$$Ku^2 + Nu - v + P = 0 \quad ,$$

$$(2Ku + N)u_1 + (-1)v_1 = 0\,.$$

By solving these equations we obtain analytical formulas for unknowns k and d. Fig. 2 (a) shows the corresponding regions for the circle and parabola considered and Fig. 2(b) the open loop Nyquist plot for the feasible point (k, k_i) with the maximal value of k_i.



(a)



(b)

Figure 2. (a) Robustness regions for $M_S = 2.0$ and parabola P. The full lines correspond to the $M_S = 2.0$ and the dashed lines correspond to the parabola P. The chosen point (marked with "+") is $k = 0.3099,\ k_i = 0.4707,\ k_d = 0.0510$; (b) Nyquist plot for the point "+".

Example 2. To illustrate the design of a robust PID controller, consider process models

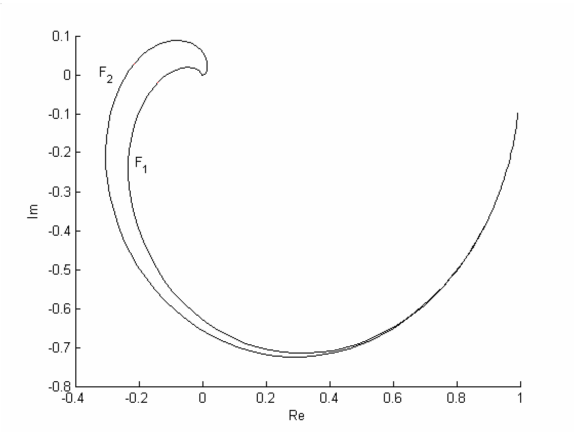
$$F_1(s) = \frac{1}{(0.2s+1)(0.4s+1)^2}\ ,\qquad F_2(s) = \frac{1}{(0.0864s+1)^5(0.5681s+1)}\ . \tag{23}$$

and the maximum sensitivity requirement $M_S = 2.0$.

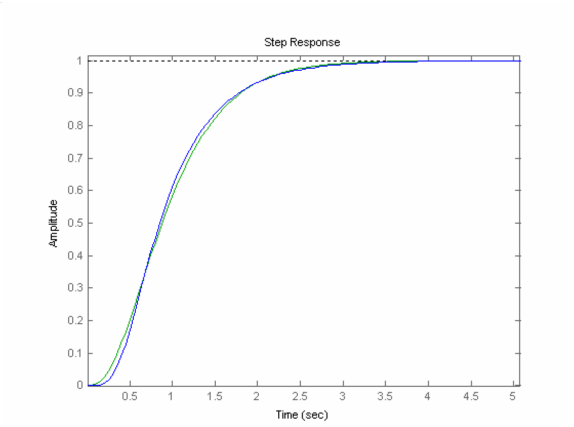
The Fig. 3 (a) shows the frequency responses of the systems (23). It is required to design robust PID controller for both transfer functions (23). The Fig. 4 shows the robustness regions for the systems $F_1(s), F_2(s)$ in the parameter plane $k - k_i$. We choose the points $A\!: k = 2.1559,\ k_i = 3.7276,\ k_d = 0.3117$ and

B: $k = 11.9404$, $k_i = 14.1113$, $k_d = 2.5259$ from this parameter plane.

Fig. 5 shows Nyquist plots of systems $F_1(s)$, $F_2(s)$ controlled by PID with parameters A (the point A in Fig. 4) and parameters B (the point B).



(a)



(b)

Figure 3 Systems $F_1(s)$, $F_2(s)$: (a) The frequency responses; (b) Step responses.

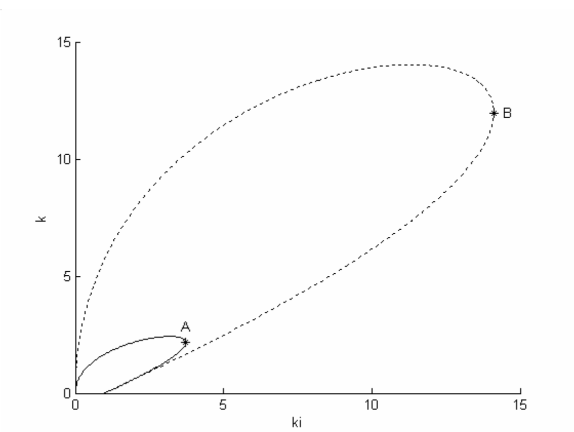
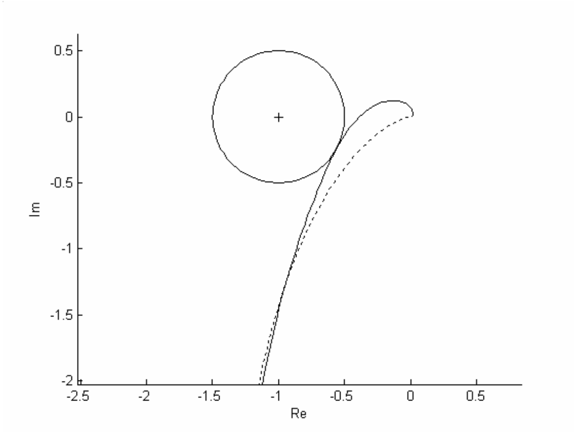
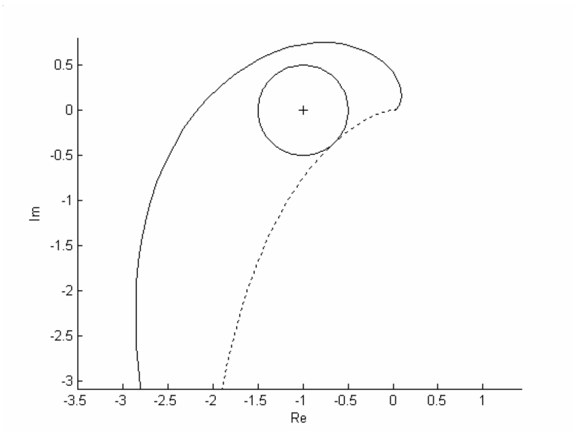


Figure 4. The regions in the parametric plane for system $F_1(s)$ is plotted full line and $F_2(s)$ dotted line.

Over the fact, that the step responses in Fig. 3 (b) are nearly the same, the regions in Fig. 4 are different. This diversity is demonstrated in Fig. 5, where in (a) is Nyquist plot for open loop with PID parameters A and in (b) with parameters B. Clearly, the system $F_1(s)$ is for PID parameters B unstable.



(a)



(b)

Figure 5 Nyquist plots (a) the parameters A; (b) the parameters B.

4 CONCLUSION

It is shown that this method can be used to design a fixed robust controller for several stable or unstable linear systems of arbitrary order with possible non-minimum-phase and irrational characteristics and including significant time delay. The possibility of the shaping open loop Nyquist plot by different conics seems to be very useful. It is suggested that the proposed technique may suit a large range of applications in industrial practice.

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