DESIGN OF ROBUST PID CONTROLLER FROM PROCESS STATIC GAIN AND ONE POINT OF FREQUENCY RESPONSE

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Abstract: The first aim of this paper is to present an explicit parametrization of the value set boundary for a process-control-oriented model family consistent with the process gain and one point of the process frequency response obtained from a relay experiment. The second aim is to develop new PID design rules based on this parameterization. Moreover, the design procedure is converted to the suitable form for autotuning purposes.

Keywords: Frequency response methods, Fractal systems, Uncertain linear systems, frequency domains, PID control design.

1. INTRODUCTION

It is well known from the classical control theory that process controllers can be designed on the base of a few points of the process frequency response. In this direction, the limit method is the popular Ziegler-Nichols frequency method which uses only one so called ultimate point. However, these traditional methods are neither systematic nor guarantee fulfillment of design specifications for an exactly given class of process transfer functions.

At present, when designing a robust controller in the frequency domain, it is usually assumed that the frequency response of the "true" system lies for each frequency on regions of the complex plane Rotstein *et al.* (1998). These regions, called value sets, quantify the amount of model uncertainty at a given frequency. The problem of obtaining hard bounds of the value sets from experimental data has been considered before, see e.g. Helmicki *et al.* (1991); Milanese *et al.* (1996). Although some of these methods are very sophisticated and general, none of them gives tight bounds for the case where only one or two frequency response points are available because of the minimum *a priori* information assumed.

In this paper, a novel approach to the above problem is presented. The key feature in the problem formulation is a new form of *a priori* information about the class of candidate models. In accordance with the majority of works in the process control field, it is assumed that the real process can be described by a multiple fractional order pole model Charef *et al.* (1992); Podlubny (1999) in the form

$$P(s) = \frac{K}{\prod_{i=1}^{p} (\tau_i s + 1)^{n_i}}, \tag{1}$$

where p is an arbitrary integer and $K, \tau_i,\ i = 1, 2, \dots, p,\ n_i,\ i = 1, 2, \dots, p$ are non-negative real numbers. It is important to note that the class of all transfer functions (1) includes all integer order lag/dead time processes because of no restriction on the total order of (1). Moreover, it follows from Skogestad (2003) that any lag/dead time transfer function with positive and/or negative numerator time constants may be also approximated (at least for the purpose of control design) by a transfer function in the form (1). Consequently, it is believed that the assumed class (1) is sufficiently rich for the process control purposes. Nevertheless, it should be explained, why the fractional order systems are introduced and preferred to common integer order transfer functions. The reason is clear from the previous paper Schlegel (2002) on the same topics, where only the integer order systems are assumed. In that case, the characterization of the value set is much more complex because of discontinuity in the pole orders. Moreover, the fractional order systems seem to be a more appropriate representation of reality.

The main result of this paper is an explicit parametrization of the value set boundary for the model family consistent with the *a priori* information (1) and two experimentally obtained points of the process frequency response. The presented results have important applications in design of robust controllers and automatic tuning procedures. For example, an exact revision of all existing PID relay autotuners can be made as outlined in Section 4.

The paper is organized as follows. The frequency response interpolation problem is formulated in Section 2. The main result of this paper dealing with one point interpolation is given in Section 3. Applications to robust controller design are outlined in Section 4. Section 5 contains concluding remarks.

2. PROBLEM FORMULATION

In this paper, a class of fractional order systems in the form (1) is considered, where $K > 0$; $\tau_i > 0$, $n_i \geq m,\ i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p} n_i \leq n$. It is assumed that m and $n,\ 0 < m < n$ are given real numbers specifying the minimum admissible order of a fractional pole and the maximum admissible order of the whole process, respectively. In the following, this class will be denoted by $\mathcal{S}^{n,m}$. Note that the span of the class $\mathcal{S}^{n,m}$ may be controlled by a proper choice of the real numbers m, n. For example, if $m = 1$ and $n \to \infty$, then the set $\mathcal{S}^{n,m}$ contains all integer order lag systems with dead time. Consequently, it is believed, that the class $\mathcal{S}^{n,m}$ is suitable for description of the *a priori* process knowledge.

A subset of $\mathcal{S}^{n,m}$ containing all transfer functions $P \in \mathcal{S}^{n,m}$ which satisfy the interpolation conditions

$$\begin{aligned} &P(\mathrm{j}\omega_i) = p_i, \\ &0 > \arg P(\mathrm{j}\omega_i) > -2\pi,\ \ i = 1, 2, \ldots, l \end{aligned} \qquad (2)$$

where $p_i \in \mathbf{C}$ and $0 \leq \omega_1 < \omega_2 < \ldots < \omega_l$ are given, is denoted by $\mathcal{S}^{n,m}(\Pi), \Pi = \{(p_i, \omega_i)\}_{i=1}^{l}$, and called the *unfalsified plant family*. Similarly, the transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is called *unfalsified*. For $l = 2$ the symbol $\mathcal{S}^{n,m}(p_1, \omega_1, p_2, \omega_2)$ will be used instead of $\mathcal{S}^{n,m}(\Pi)$. Thus, if $P \in \mathcal{S}^{n,m}(p_1, \omega_1, p_2, \omega_2)$, then the transfer function P must have the form (1) with the above constraints and its Nyquist plot $P(\mathrm{j}\omega)$ is required to begin in the point p_1 at the frequency $\omega_1 = 0$ and pass through the given point p_2 at the frequency ω_2 during the first circulation around the origin of the complex plane. Point out that the data Π are assumed to be obtained from an identification experiment. In the following, the key concept is the value set, which is defined for any ω by

$$\mathcal{V}_{\omega}^{n,m}(\Pi) = \{P(\mathrm{j}\omega) :\ P \in \mathcal{S}^{n,m}(\Pi)\}. \qquad (3)$$

The boundary of the value set $\mathcal{V}_{\omega}^{n,m}(\Pi)$ will be denoted by $\partial\mathcal{V}_{\omega}^{n,m}(\Pi)$. The main aim of this paper is to present an exact description of the boundary $\partial\mathcal{V}_{\omega}^{n,m}(\Pi)$ for the case $l = 2$. For this purpose, the concept of an ultimate transfer function is introduced.

Definition 1. The unfalsified transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is called ultimate if there exists $\omega \notin \{\omega_1, \omega_2, \ldots, \omega_l\}$ such that $P(\mathrm{j}\omega) \in \partial\mathcal{V}_{\omega}^{n,m}(\Pi)$.

In other words, $P \in \mathcal{S}^{n,m}(\Pi)$ is ultimate if $P(\mathrm{j}\omega)$ creates a boundary point of $\mathcal{V}_{\omega}^{n,m}(\Pi)$ for at least one frequency $\omega > 0$ different from interpolating frequencies $\omega_1, \omega_2, \ldots, \omega_l$. It will be shown later that if P generates a boundary point for one such frequency, then P generates a boundary point for all $\omega > 0$. Thus, in our context, the concept of an ultimate transfer function is independent of the frequency. Consequently, the parametrization of all ultimate transfer functions is sufficient for exact description of the boundary $\partial\mathcal{V}_{\omega}^{n,m}(\Pi)$.

3. TWO POINT INTERPOLATION

This section deals with the simple case when the experimental data contain only two samples of the frequency response. The next theorem gives an exact parametrization of all ultimate transfer functions for this case.

Theorem 1. Let $\Pi = \{(p_1, \omega_1), (p_2, \omega_2)\}\ ,\ \ p_1 = 1, \omega = 0,\ \ p_2 = r\mathrm{e}^{-\mathrm{j}\varphi},\ r > 0,\ 0 < \varphi < 2\pi,\ \omega_2 > 0$ and $\varphi \leq (n - m)\frac{\pi}{2}$. Then $\mathcal{S}^{n,m}(\Pi) \neq \emptyset$ and the unfalsified transfer function $P \in \mathcal{S}^{n,m}(\Pi)$ is ultimate if and only if it can be expressed in one of the following forms

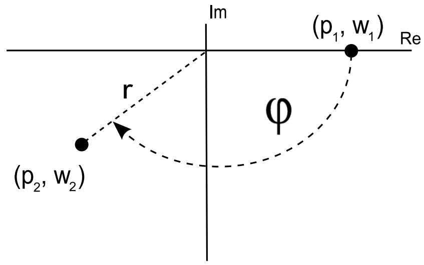

Fig. 1. The two measured samples of frequency response $(p_1 = 1, \omega_1 = 0)$ and $(p_2 = re^{-j\varphi}, \omega_2 = 1)$.

(*i*) $P_1\,(s, \alpha_1, \alpha_2) =$ (4)

$$= \frac{1}{(\tau_1(\alpha_1)s+1)^m(\tau_2(\alpha_2)s+1)^{n_{12}(\alpha_1,\alpha_2)}},$$

where

$$\tau_1(\alpha_1) \le \tau_2(\alpha_2),$$
$$\tau_1(\alpha_1) = \tan\alpha_1,$$
$$\tau_2(\alpha_2) = \tan\alpha_2,$$
$$n_{12}(\alpha_1, \alpha_2) = \frac{\varphi - m\alpha_1}{\alpha_2}$$

and the parameters α_1 and α_2 fulfill the following conditions:

$$\alpha_1 \in I_{11},$$
$$\alpha_2 \in I_{12},$$
$$\sqrt{(\tan^2\alpha_1 + 1)^m(\tan^2\alpha_2+1)^{n_{12}}} = \frac{1}{r}$$

where

$$I_{11} = \left[max\left\{0, \frac{\varphi - (n-m)\frac{\pi}{2}}{m}\right\}, min\left\{\frac{\pi}{2}, \frac{\varphi}{2m}\right\}\right]$$

$$I_{12} = \left[\max\left\{0, \alpha_1, \frac{\varphi - m\alpha_1}{n-m}\right\}, \min\left\{\frac{\pi}{2}, \frac{\varphi}{m} - \alpha_1\right\}\right]$$

(*ii*) $P_2\,(s, \alpha_1, \alpha_2) =$ (5)

$$= \frac{1}{(\tau_1(\alpha_1)s+1)^{n_{21}(\alpha_1,\alpha_2)}(\tau_2(\alpha_2)s+1)^m},$$

where

$$\tau_1(\alpha_1) \le \tau_2(\alpha_2),$$
$$\tau_1(\alpha_1) = \tan\alpha_1,$$
$$\tau_2(\alpha_2) = \tan\alpha_2,$$
$$n_{21}(\alpha_1, \alpha_2) = \frac{\varphi - m\alpha_2}{\alpha_1}$$

and the parameters α_1 and α_2 fulfill the following conditions:

$$\alpha_1 \in I_{11},$$
$$\alpha_2 \in I_{12},$$
$$\sqrt{(\tan^2\alpha_1 + 1)^{n_{21}}(\tan^2\alpha_2+1)^{m}} = \frac{1}{r}$$

where

$$I_{21} = \left[max\left\{0, \frac{\varphi - m\frac{\pi}{2}}{n-m}\right\}, min\left\{\frac{\pi}{2}, \frac{\varphi}{2m}\right\}\right]$$

$$I_{22} = \left[\max\left\{0, \alpha_1, \frac{\varphi - (n-m)\alpha_1}{m}\right\}, \min\left\{\frac{\pi}{2}, \frac{\varphi}{m} - \alpha_1\right\}\right]$$

(*iii*) $P_3\,(s, \alpha_1, \alpha_2) =$ (6)

$$= \frac{1}{(\tau_1(\alpha_1)s+1)^{n-n_{32}(\alpha_1,\alpha_2)}(\tau_2(\alpha_2)s+1)^{n_{32}(\alpha_1,\alpha_2)}},$$

where

$$\tau_1(\alpha_1) \le \tau_2(\alpha_2),$$
$$\tau_1(\alpha_1) = \tan\alpha_1,$$
$$\tau_2(\alpha_2) = \tan\alpha_2,$$
$$n_{32}(\alpha_1, \alpha_2) = \frac{\varphi - n\alpha_1}{\alpha_2 - \alpha_1}$$

and the parameters α_1 and α_2 fulfill the following conditions:

$$\alpha_1 \in I_{11},$$
$$\alpha_2 \in I_{12},$$
$$\sqrt{(\tan^2\alpha_1 + 1)^{n-n_{32}}(\tan^2\alpha_2+1)^{n_{32}}} = \frac{1}{r}$$

where

$$I_{31} = \left[max\left\{0, \frac{\varphi - (n-m)\frac{\pi}{2}}{m}\right\}, min\left\{\frac{\pi}{2}, \frac{\varphi}{m}\right\}\right]$$

$$I_{32} = \left[\max\left\{0, \alpha_1, \frac{\varphi - m\alpha_1}{n-m}\right\}, \min\left\{\frac{\pi}{2}, \frac{\varphi - (n-m)\alpha_1}{m}\right\}\right]$$

Moreover, the value set $\mathcal{V}_\omega^{n,m}(\Pi)$, $\omega > 0$, $\omega \neq \omega_1$ and $\omega \neq \omega_2$ is a closed domain bounded by three arcs: $P_1(\mathrm{j}\omega, \alpha_1, \alpha_2)$, $\alpha_1 \in I_{12}$, $\alpha_2 \in I_{21}$; $P_2(\mathrm{j}\omega, \alpha_1, \alpha_2), \alpha_1 \in I_{21}, \alpha_2 \in I_{22}$ and $P_3(\mathrm{j}\omega, \alpha_1, \alpha_2)$, $\alpha_1 \in I_{31}, \alpha_2 \in I_{32}$. □

Several remarks can be made regarding Theorem 1.

Remark 2. The version of the Theorem 1 for the case where no *a priori* restriction on the process order is considered, can be simply obtained by the limiting process $n \to \infty$.

Remark 3. For the detail analysis of the value set $\mathcal{V}_\omega^{n,m}(1, 0, r\mathrm{e}^{-\mathrm{j}\varphi}, \omega_2)$, we can restrict ourselves without loss of generality to the case $0 < r \le 1$ and $\omega_2 = 1$ because of the normalization in gain and time. Consequently, all illustrating figures given below will be for this case (Fig. 2 and 3).

Remark 4. Hereafter, the endpoints of the arcs $P_i(\mathrm{j}\omega, \alpha_1, \alpha_2)$, $\alpha_1 \in I_{i1}$, $\alpha_2 \in I_{i2}$, $i = 1, 2, 3$, are called vertices of the value set $\mathcal{V}_\omega^{n,m}(p_1, \omega_1, p_2, \omega_2)$ (see Fig. 2). It follows from Theorem 1 that they are generated by the transfer functions

$$V_1(s) = \lim_{\alpha_1 \to \alpha_{11}^-} P_1(s, \alpha_1, \alpha_2) = \lim_{\alpha_1 \to \alpha_{31}^-} P_3(s, \alpha_1, \alpha_2),$$
$$V_2(s) = \lim_{\alpha_1 \to \alpha_{11}^+} P_1(s, \alpha_1, \alpha_2) = \lim_{\alpha_1 \to \alpha_{21}^+} P_2(s, \alpha_1, \alpha_2),$$
$$V_3(s) = \lim_{\alpha_1 \to \alpha_{21}^-} P_2(s, \alpha_1, \alpha_2) = \lim_{\alpha_1 \to \alpha_{31}^+} P_3(s, \alpha_1, \alpha_2),$$

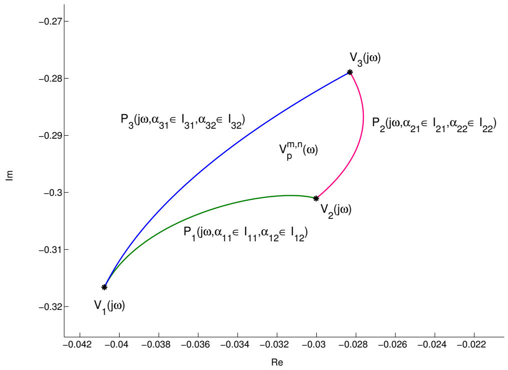

Fig. 2. The value set $\mathcal{V}_{0.7}^{50,1}(1, 0, 0.3\ \mathrm{e}^{-\mathrm{j}1.92}, 1)$.

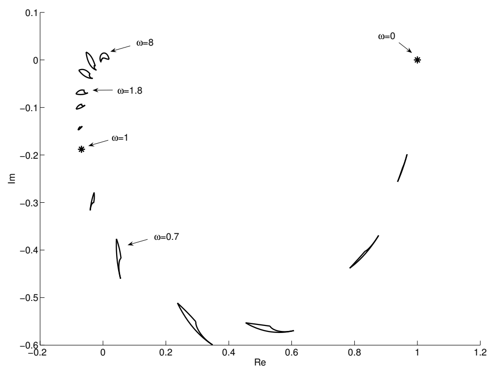

Fig. 3. The value sets $\mathcal{V}_{\omega}^{50,1}(1, 0, 0.3\ \mathrm{e}^{-\mathrm{j}1.92}, 1)$ for several ω.

where α_{i1}^{-} is the minimal value and α_{i1}^{+} is the maximal value, respectively, from the interval I_{i1} for which exists $\alpha_{i2} \in I_{i2}$ that it is fulfill the following equation

$$\sqrt{(\tan^2 \alpha_1 + 1)^{n_{i1}} (\tan^2 \alpha_2 + 1)^{n_{i2}}} = \frac{1}{r},$$

where

$$n_{11} = m, \qquad n_{12} = \frac{\varphi - n\alpha_1}{\alpha_2 - \alpha_1},$$
$$n_{21} = \frac{\varphi - m\alpha_2}{\alpha_1}, \quad n_{22} = m,$$
$$n_{31} = n - n_{32}, \qquad n_{32} = \frac{\varphi - n\alpha_1}{\alpha_2 - \alpha_1}.$$

4. ROBUST CONTROLLER DESIGN

In this section, Theorem 1 and the robustness regions method Shafiei and Shenton (1997); Astrom and Hagglund (2001) will be employed in design of the robust PID controller when the process gain p_1 and only one point (p_2, ω_2) of the process frequency response are available. Suppose that $p_1 = 1, \omega_1 = 0, p_2 = 0.5\ \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$ and $\omega_2 = 1$. Then, according to Theorem 1 and Remark 4, the value set $\mathcal{V}_{\omega}^{50,1}(1, 0, 0.5\ \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}, 1)$ has only the two

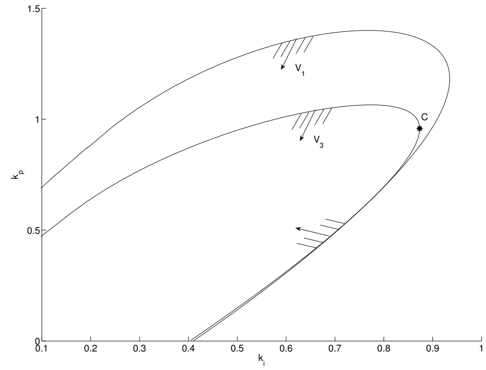

Fig. 4. The robustness regions for vertex processes.

vertices. Further, the PID controller is assumed in the form

$$C(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{\alpha s + 1} = \\ = K + \frac{1}{T_i s} + \frac{T_d s}{\frac{T_d}{N} s + 1}, \qquad (7)$$

where K, T_i, and T_d are its engineering parameters and $k_p, k_i = k_p/T_i, k_d = kT_d$ are gains used in the following. We restrict the problem to the case of two design parameters k and k_i and fix the ratio $f = T_d/T_i = 1/4$ and the filtr parameter $N = 10$.

The first step is to design the parameters k, k_i such that the sensitivity

$$M_s = sup_\omega |S(j\omega)|$$

where $S(s)$ is the sensitivity function of open loop transfer function, is at least $M_s = 2$ for both vertices process transfer functions. Corresponding robustness regions in the parameter plane $k - k_i$ are depicted in Fig. 4. Any point from their intersection (the inner region in Fig. 4) fulfills the above design specification $M_s = 2$. Moreover, the point with the maximum k_i coordinate minimizes the criterion $\int_0^\infty e(t)dt$ Astrom and Hagglund (2001). Choosing this point (the point C in Fig. 4), $k = 0.87,\ k_i = 0.96$. To complete the robust design procedure, it must be checked whether all unfalsified transfer functions given by Theorem 1 also fulfill the design specifications. However, in our case this fact is apparent from Fig. 5.

New PID design rules can be developed based on the combination of the parametrization of the vertices from Section 3 and robustness regions method for design PID controllers. We decided to compute PID controller gains k_i and k_p for all pairs (r, φ) which correspond to the measured points $(p_2 = re^{-j\varphi}, \omega_2 = 1)$ of frequency response. Parameter φ is from interval $[\pi/2, \pi]$ with the step 0.0873 and distance from the origin r is from $[0.1, 0.9]$ with the step 0.05. The results are depicted in Fig.6 and Fig.7 and moreover they

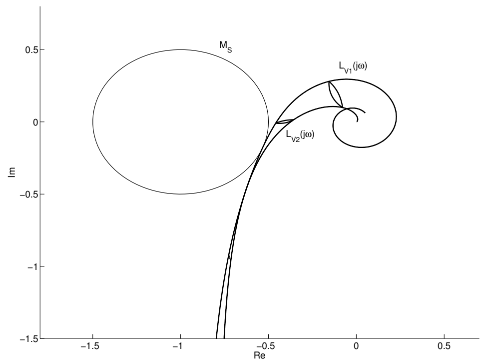

Fig. 5. The value sets compensated by the designed PID controller.

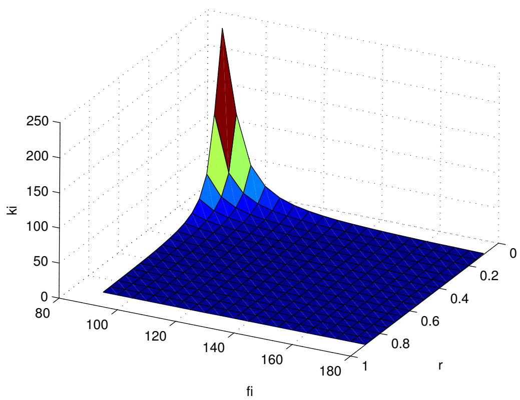

Fig. 6. The computed values of k_p in different points (r, φ).

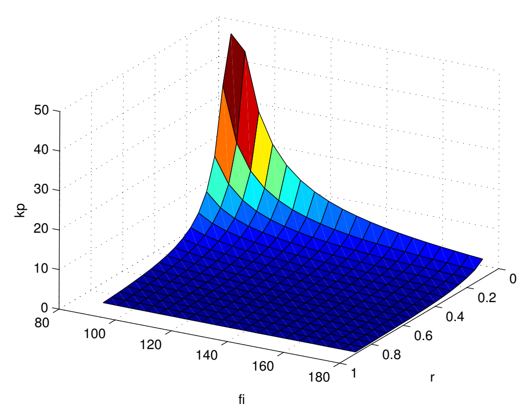

Fig. 7. The computed values of k_i in different points (r, φ).

are prepared for conversion to suitable form for autotunning purposes.

5. CONCLUSIONS

In this paper, an explicit parametrization of the value set boundary for a multiple fractional order pole model has been presented under the assumption that only process gain and one point of the process frequency response are available. In particular, it is shown that the value sets are closed domains bounded by at most three smooth arcs. The presented results have important applications in design of robust controllers, in particular for automatic tuning procedures based on one or two points of a process frequency response.

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